Homomorphisms from AH-algebras

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Abstract

Let C be a general unital AH-algebra and let A be a unital simple C^*-algebra with tracial rank at most one. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. We show that φ and ψ are approximately unitarily equivalent if and only if

$$\begin{aligned} [\varphi] &= [\psi] \text{ in } KL(C,A), \\ \varphi_\sharp &= \psi_\sharp \text{ and} \\ \varphi^\dagger &= \psi^\dagger, \end{aligned} \tag{e 0.1}$$

where $\varphi_\sharp$ and $\psi_\sharp$ are continuous affine maps from tracial state space $T(A)$ of A to faithful tracial state space $T_{\rm f}(C)$ of C induced by φ and ψ, respectively, and $\varphi^\ddagger$ and $\psi^\ddagger$ are induced homomorphisms from $K_1(C)$ into $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$, where $\mathrm{Aff}(T(A))$ is the space of all real affine continuous functions on $T(A)$ and $\rho_A(K_0(A))$ is the closure of the image of $K_0(A)$ in the affine space $\mathrm{Aff}(T(A))$. In particular, the above holds for $C = C(X)$, the algebra of continuous functions on a compact metric space. An approximate version of this is also obtained. We also show that, given a triple of compatible elements $\kappa \in KL_e(C,A)^{++}$, an affine map $\gamma : T(C) \to T_{\rm f}(C)$ and a homomorphism $\alpha : K_1(C) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$, there exists a unital monomorphism $\varphi : C \to A$ such that $[h] = \kappa$, $h_\sharp = \gamma$ and $\varphi^\dagger = \alpha$.

1 Introduction

Let X be a compact metric space and let A be a unital simple C^*-algebra. Let φ, $\psi : C(X) \to A$ be two homomorphisms. We study the problem when these two maps from $C(X)$, the commutative C^*-algebra of continuous functions on X, into A are approximately unitarily equivalent, i.e., when there exists a sequence of unitaries $\{u_n\} \subset A$ such that

$$\lim_{n\to\infty} u_n^* \psi_n(f) u_n = \varphi(f) \text{ for all } f \in C(X).$$

In the case that X is a compact subset of the plane and A is the $n \times n$ matrix algebra, two such maps are unitarily equivalent if and only if the corresponding normal matrices have the same set of eigenvalues (counting multiplicity). Brown-Douglass-Fillmore's study of essentially normal operators led to the following theorem: Two unital monomorphisms from $C(X)$ (when X is a compact subset of the plane) into the Calkin algebra are unitarily equivalent if and only if they induce the same homomorphism from $K_1(C(X))$ into $\mathbb{Z}$. It should be noted that both the $n \times n$ matrix algebra and the Calkin algebra are unital simple C^*-algebras of real rank zero.

Unital separable commutative C^*-algebras are of the form $C(X)$ for some compact metric space by the Gelfand transformation. Therefore the study of C^*-algebras may be viewed as the study of non-commutative topology. As in the topology, one studies continuous maps between spaces, in C^*-algebra theory, one studies the homomorphisms from one C^*-algebra to another. In this point view, the study of homomorphisms from one C^*-algebra to another is one of the fundamental problems in the C^*-algebra theory. At the present paper, we assume that the target algebra is a unital simple C^*-algebra, which conforms to the previous two mentioned cases. Simple C^*-algebras may also be viewed as the opposite end of commutative C^*-algebras.

For the source algebra, we begin with the case that it is the commutative C^*-algebra following the two above mentioned cases. However, we will study the case that source algebras are general unital AH-algebras (They are not necessarily simple, nor of slow dimension growth).

Let $\varphi, \psi : C(X) \to A$ be two unital homomorphisms and let $I = \ker\varphi$. Then $I = \ker\psi$, if φ and ψ are approximately unitarily equivalent. Therefore, one may study the induced homomorphisms from $C(X)/I$ instead. Note that $C(X)/I$ is isomorphic to $C(Y)$ for some compact subset of X. To simplify the matter, we will only study monomorphisms. The problem has been studied (for some earlier results, for example, see [8] and [9]). Dadarlat ([2]) showed that, if $C = C(X)$ and A is a unital purely infinite simple C^*-algebra (such as Calkin algebra), then two unital monomorphisms from C into A are approximately unitarily equivalent if and only if they induce the same element in $KL(C, A)$. When the target C^*-algebras are finite, other invariants such as traces have to be considered. When A is a unital simple C^*-algebra with stable rank one, real rank zero, weakly unperforated $K_0(A)$ and a unique tracial state, it is shown in [6] that φ and ψ are approximately unitarily equivalent if and only if $[\varphi] = [\psi]$ in $KL(C(X), A)$ and $\tau \circ \varphi = \tau \circ \psi$. When the real rank of A is not zero one needs additional data to determine when φ and ψ are approximately unitarily equivalent. In fact, it is shown ([18]) that when C is a some special unital AH-algebra and A is a unital simple C^*-algebra with tracial rank at most one, two unital monomorphisms $\varphi, \psi : C \to A$ are approximately unitarily equivalent if and only $[\varphi] = [\psi]$, $\varphi_\sharp = \psi_\sharp$ and $\varphi^\ddagger = \psi^\ddagger$, where $\varphi_\sharp$ and $\varphi^\ddagger$ will be defined below (2.1) and (2.3). The technical condition imposed on $C(X)$ is basically said that, K-theoretically speaking, $C(X)$ has a lower rank. In this paper this restriction on AH-algebras has been removed. A complete criterion is given for two unital monomorphisms from a general AH-algebra into a unital simple C^*-algebra with tracial rank at most one being approximately unitarily equivalent.

One may view the result of this paper is a generalization of that in [18]. However, this generalization have a number important applications. First, the improvement is based on the proof of Theorem 3.6 below. The proof of the main result in [18] among many things uses Theorem 3.2 of [18] which in turn, among other things, used the technical decomposition theorem of Guihua Gong ([5]). Gong's theorem has a very technical and long proof. The proof of this paper does not require to use Gong's decomposition theorem. Gong's decomposition theorem played the key role in the classification of unital simple AH-algebras with no dimension growth ([4]). While the classification theorem for unital simple separable amenable C^*-algebras with tracial rank at most one satisfying the UCT in [15] do not require Gong's theorem, however, it is Gong's decomposition theorem which shows that every unital simple AH-algebras with very slow dimension growth have tracial rank at most one. As in [18], one sees that the main result of this paper can be used to provide a proof of classification theorem for unital simple AH-algebras with slow dimension growth. Therefore, one can now provide a proof of classification theorem of unital simple AH-algebras with slow dimension growth without using the celebrated Gong's decomposition theorem ([21]).

There are much more than just shorten the proof. One of the long standing problems in the classification theory is to classify locally AH-algebra with no dimension growth. The problem could be solved if one could establish a version of Gong's decomposition theorem which allows maps that are not exactly homomorphisms. Over more than a decade, since the proof Gong's decomposition theorem first appeared, the technical difficulty to generalize it to include almost multiplicative maps had remained elusive. This author's many attempts failed during these years. It is the desire to prove that unital simple locally AH-algebras with no dimension growth can be classified by their Elliott invariant drew author's attention again to Gong's decomposition theorem. One application of the results in this paper will be the proof that unital simple locally AH-algebras with slow dimension growth are classifiable by the Elliott invariant ([21]).

Having stated the importance of the results in this paper in the connection of the Elliott pro-

gram of classification of amenable C^*-algebras and their independence of Gong's decomposition theorem, making no mistake, however, we did *not* provide another proof of Gong's decomposition theorem, nor we provide a version of Gong's decomposition theorem working for almost multiplicative maps. Instead, we establish a so-called uniqueness theorem for almost multiplicative maps from unital AH-algebras to unital simple C^*-algebra with tracial rank at most one (Theorem 5.3). Even without referring to Gong's decomposition theorem and classification of simple amenable C^*-algebras, we believe that the main results presented here have their own independent interest as discussed at the beginning of this introduction.

The paper is organized as follows: Section 2 serves largely as preliminaries for the whole paper. In Section 3, we prove Theorem 3.6 which is the main technical advance of this paper. In Section 4, we collect a number of miscellaneous lemmas which will be used in the proof of the main results. In Section 5, we prove the main results. To complete our results and make application possible. In Section 6, we provide the description of range of approximate unitary equivalence classes of unital monomorphisms from a unital AH-algebra to a unital simple C^*-algebra of tracial rank at most one. Applications to the study of tracial rank and classification of unital simple locally AH-algebras will appear elsewhere ([21]).

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2 Preliminaries

2.1. Let A be a unital C^*-algebra. Denote by $T(A)$ the convex set of tracial states of C. Denote by $T_{\rm f}(A)$ the convex set of all faithful tracial states. Let $\mathrm{Aff}(T(A))$ be the space of all real affine continuous functions on $T(A)$. Denote by $M_n(A)$ the matrixes over A. By regarding $M_n(A)$ as a subset of $M_{n+1}(A)$, define $M_\infty(A) = \cup_{n=1}^\infty M_n(A)$. If $\tau \in T(A)$, then $\tau \otimes Tr$, where tr is standard trace on M_n, is a trace on $M_n(A)$. Throughout this paper, we will use τ for $\tau \otimes Tr$ without warning.

If B is another C^*-algebra and $\varphi : A \to B$ be a contractive completely positive linear map, then $\varphi \otimes \mathrm{id}_{M_n}$ gives a contractive completely positive linear map from $M_n(A)$ to $M_n(B)$. Throughout this paper, we will use φ for $\varphi \otimes \mathrm{id}_{M_n}$ for convenience.

Let C and A be two unital C^*-algebras with $T(C) \neq \emptyset$ and $T(A) \neq \emptyset$. Suppose that $h : C \to A$ is a unital homomorphism. Define an affine continuous map $h_\sharp : T(A) \to T(C)$ by $h_\sharp(\tau)(c) = \tau \circ h(c)$ for all $\tau \in T(A)$ and $c \in C$. If A is simple and h is a monomorphism, then $h_\sharp$ maps $T(A)$ into $T_{\rm f}(C)$.

Definition 2.2. Let C be a unital C^*-algebra with $T(C) \neq \emptyset$. For each $p \in M_n(C)$ define $\check{p}(\tau) = \tau \otimes \mathrm{Tr}(p)$ for all $\tau \in T(A)$, where Tr is the standard trace on M_n. This gives positive homomorphism $\rho_C : K_0(C) \to \mathrm{Aff}(T(C))$.

2.3. Let C be a unital C^*-algebra. Denote by $U(C)$ the unitary group of C and denote by $U_0(C)$ the subgroup of $U(C)$ consisting of unitaries which connected to 1_C by a continuous path of unitaries. Denote by $CU(C)$ be the closure of the normal subgroup generated by commutators of $U(C)$. Let $u \in U(C)$. Then $\bar{u}$ is the image of u in $U(C)/CU(C)$. Denote by $CU_0(C)$ the intersection $CU(C) \cap U_0(C)$.

Now suppose that $T(C) \neq \emptyset$. Let $n \geq 1$ be an integer. Let $u \in U_0(M_n(C))$. Let $\gamma \in$

$C([0,1], U_0(M_n(C)))$ which is piecewise smooth such that $\gamma(1) = u$ and $\gamma(0)$. Define

$$\Delta(\gamma)(\tau) = \int_0^1 \tau(\frac{d\gamma(t)}{dt}\gamma(t))dt, \tag{e 2.2}$$

where τ is identified with $\tau \otimes Tr$ (note that, for $n > 1$, τ in the above formula is not the normalized trace). As in K. Thomsen (see [24]), the de la Harp-Scandalis determinant provides a continuous homomorphism

$$\bar{\Delta} : \cup_{k=1}^{\infty}(U_0(M_n(C))/U_0(M_n(C)) \cap CU_0(M_k(C)) \to \mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))}. \tag{e 2.3}$$

We will use $CU(M_\infty(C))$ for $\cup_{k=1}^{\infty} CU_0(M_k(C))$.

Define a metric as follows. If $u, v \in U(M_n(C))$ such that $uv^* \in U_0(M_n(C))$, define

$$\mathrm{dist}(\bar{u}, \bar{v}) = \|\bar{\Delta}(uv^*)\|. \tag{e 2.4}$$

Note that if $u, v \in U_0(M_n(C))$, then

$$\mathrm{dist}(\bar{u}, \bar{v}) = \|\bar{\Delta}(u) - \bar{\Delta}(v)\|$$

(where the norm is the quotient norm in $\mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))}$).

Note that if $u \in CU(C)$, then $[u] = 0$ in $K_1(C)$. Using de la Harp-Scandalis determinant, by K. Thomsen (see [24]), one has the following short splitting exact sequence:

$$0 \to \mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))} \to U(M_\infty(C))/CU(M_\infty(C)) \to K_1(C) \to 0. \tag{e 2.5}$$

We will fix one splitting map $J_C : K_1(C) \to U(M_\infty(C))/CU(M_\infty(C))$. For each $\bar{u} \in J(K_1(C))$, select and fix one element $u_c \in \cup_{n=1}^{\infty} M_n(C)$ such that $\overline{u_c} = \bar{u}$. Denote this set by $U_c(K_1(C))$.

If A is a unital C^*-algebra and $\varphi : C \to A$ is a unital homomorphism, then φ induces a continuous map

$$\varphi^{\ddagger} : U(M_\infty(C))/CU(M_\infty(C)) \to U(M_\infty(A))/CU(M_\infty(A)).$$

Denote by $\varphi^{\dagger} : K_1(C) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ the map $(\mathrm{id} - J_A) \circ \varphi^{\ddagger} \circ J_c$, where $J_A : K_1(A) \to U(M_\infty(A))/CU(M_\infty(A)$ is a fixed splitting map.

If $K_1(C) = U(C)/U_0(C)$, then, by [24],

$$U_0(C)/CU(C) = U_0(M_n(C))/CU(M_n(C))$$

for all $n \geq 1$.

2.4. Let A be a unital C^*-algebra and let $u \in U_0(A)$. Let $\gamma \in C([0,1], U(A))$ such that $\gamma(0) = 1$ and $\gamma(1)$. Denote by $\mathrm{Length}(\{\gamma\})$ the length of the path γ. Put

$$\mathrm{cel}(u) = \inf\{\mathrm{Length}\gamma(u) : \gamma \in C([0,1], U(A)), \gamma(0) = 1 \text{ and } \gamma(1) = u\}.$$

Definition 2.5. Let C be a C^*-algebra and let $\mathcal{P} \subset \underline{K}(C)$. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ such that, for any δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C \to A$ (for any C^*-algebra A), $[L]|_{\mathcal{P}}$ is well defined (see 0.6 of [10] and 2.3 of [17]). Such a triple $(\delta, \mathcal{G}, \mathcal{P})$ is called local $\underline{K}$-triple (see [3]). If $K_i(C)$ is finitely generated $(i = 0, 1)$ and $\mathcal{P}$ is large enough, then $[L]|_{\mathcal{P}}$ defines an element in $KK(C, A)$ (see 2.4 of [17]). In such cases, we will write $[L]$ instead of $[L]|_{\mathcal{P}}$, and we will call $(\delta, \mathcal{G}, \mathcal{P})$ a KK-triple and $(\delta, \mathcal{G})$ a KK-pair. Note that, if u is a unitary then, we write $\langle L(u)\rangle = L(u)(L(u)^*L(u))^{-1/2}$ when $\|L(u^*)L(u) - 1\| < 1$

and $\|L(u)L(u^*)-1\|<1$. In what follows we will always assume that $\|L(u^*)L(u)-1\|<1$ and $\|L(u)L(u^*)-1\|<1$, when we write $\langle L(u)\rangle$.

Suppose that C is a separable C^*-algebra and C is the closure of $\cup_{n=1}^{\infty}C_n$, where each $C_n=\lim_{m\to\infty}(C_{n,m},\varphi_m^{(n)})$ and $K_i(C_{n,m})$ is finitely generated $(i=0,1)$. Denote by $\imath_n: C_n\to C$ the embedding and $\varphi_{m,\infty}^{(n)}: C_{n,m}\to C_n$ the homomorphism induced by the inductive system $(C_{n,m},\varphi_m)$. We say that $(\delta,\mathcal{G},\mathcal{P})$ is a KL-triple, $[\imath_n\circ\varphi_{m,\infty}^{(n)}](\mathcal{P}')\supset\mathcal{P}$ for some n, m and some finite subset $\mathcal{P}'\subset\underline{K}(C_{n,m})$ and if any δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L: C\to A$ (for any A), gives a δ-$\mathcal{G}'$-multiplicative contractive completely positive linear map $L\circ\imath_n\circ\varphi_{m,\infty}^{(n)}$ so that $(\delta,\mathcal{G}',\mathcal{P}')$ is a KK-triple.

2.6. If A is a unital C^*-algebra with tracial rank at most one, then we will write $TR(A)\le 1$ (see [12]).

Definition 2.7. Let X be a compact metric space, let $x\in X$ and let $r>0$. Denote by $O(x,r)$ the open ball with center at x and radius r. If x is not specified $O(r)$ is an open ball of radius r.

The following could be proved directly but also follows from 4.6 of [14].

Theorem 2.8. *Let X be a compact metric space, let $\epsilon>0$ and let $\mathcal{F}\subset C(X)$ be a finite subset. There exists $\eta>0$ satisfying the following: for any $\sigma>0$, there exists $\gamma>0$, $\delta>0$, a finite subset $\mathcal{G}\subset C(X)$ and a finite subset $\mathcal{H}\subset C(X)_{s.a}$ and a finite subset $\mathcal{P}\subset\underline{K}(C(X))$ satisfying the following:*

For any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $\varphi,\psi: C(X)\to M_n$ (for some integer $n\ge 1$) for which

$$[\varphi]|_{\mathcal{P}}=[\psi]|_{\mathcal{P}},\ \ \mu_{\tau\circ\varphi}(O_r)\ge\sigma \tag{e 2.6}$$

for all open balls O_r of radius $r\ge\eta$ and

$$|\tau\circ\varphi(a)-\tau\circ\psi(a)|<\gamma\ \text{ for all }\ a\in\mathcal{H}, \tag{e 2.7}$$

there is a unitary $u\in M_n$ such that

$$\|\varphi(f)-\mathrm{Ad}\,u\circ\psi(f)\|<\epsilon\ \text{ for all }\ f\in\mathcal{F}. \tag{e 2.8}$$

The following is an variation of Lemma 4.3 of [18].

Corollary 2.9. *Let X be a compact metric space, $\epsilon>0$ and $\mathcal{F}\subset C(X)$ be a finite subset. There exists $\eta_1>0$ satisfying the following: for any $\sigma_1>0$ and any $0<\lambda<1$, there exists $\eta_2>0$ satisfying the following: for any $\sigma_2>0$, there exists $\delta>0$, a finite subset $\mathcal{G}\subset C(X)$ and a finite subset $\mathcal{P}\subset\underline{K}(C(X))$ satisfying the following:*

For any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\varphi: C(X)\to M_n$ (for some integer $n\ge 1$) such that

$$[\varphi]|_{\mathcal{P}}=[H]|_{\mathcal{P}} \tag{e 2.9}$$

for some unital homomorphism $H: C(X)\to M_n$ and such that

$$\mu_{\tau\circ\varphi}(O_r)\ge\sigma_1\ \text{ and }\ \mu_{\tau\circ\varphi}(O_r)\ge\sigma_2 \tag{e 2.10}$$

for all open balls O_r of radius $r\ge\eta_1$ and $r\ge\eta_2$, respectively, there is a unital homomorphism $h: C(X)\to M_n$ such that

$$\|\varphi(f)-h(f)\|<\epsilon\ \text{ for all }\ f\in\mathcal{F}. \tag{e 2.11}$$

Moreover,

$$\mu_{troh}(O_r) \ge \lambda\sigma_1 \tag{e 2.12}$$

for all $r \ge 2\eta_1$.

2.10. Let C be a unital C^*-algebra and let $\mathcal{P} \subset \underline{K}(C)$ be a finite subset. There is a finite subset $\mathcal{F}_{C,\mathcal{P},\mathbf{b}} \subset C$ and a positive number $\delta_{C,\mathcal{P},\mathbf{b}} > 0$ such that $\mathrm{Bott}(u,\, h)|_{\mathcal{P}}$ (see the definition 2.10 of [17]) is well defined for any unital C^*-algebra A, any unital homomorphism $h : C \to A$ and any unitary $u \in A$ for which

$$\|[h(f),\, u]\| < \delta_{C,\mathcal{P},\mathbf{b}}. \tag{e 2.13}$$

Moreover, by choosing even smaller $\delta_{C,\mathcal{P},\mathbf{b}}$, if $h_1 : C \to A$ is another unital homomorphism and

$$\|h(f) - h_1(f)\| < \delta_{C,\mathcal{P},\mathbf{b}},$$

then $\mathrm{Bott}(u,\, h_1)|_{\mathcal{P}}$ is also well defined and

$$\mathrm{Bott}(u,\, h)|_{\mathcal{P}} = \mathrm{Bott}(u,\, h_1)|_{\mathcal{P}}.$$

As in tradition,

$$\mathrm{bott}_1(u,\, h)|_{\mathcal{P}} = \mathrm{Bott}(u,\, h)|_{\mathcal{P}\cap K_1(C)} \ \text{ and } \ \mathrm{bott}_0(u,\, h)|_{\mathcal{P}} = \mathrm{Bott}(u\, h)|_{\mathcal{P}\cap K_0(C)}.$$

If $K_i(C)$ $(i = 0, 1)$ is finitely generated, then, by choosing $\mathcal{P}$ large enough, we may assume that, when (e 2.13) holds, $\mathrm{Bott}(h,\, u)$ is well defined. Furthermore, we will write $\delta_{C,\mathbf{b}}$ instead of $\delta_{C,\mathcal{P},\mathbf{b}}$ and $\mathcal{F}_{C,\mathbf{b}}$ instead of $\mathcal{F}_{C,\mathcal{P},\mathbf{b}}$.

If $C = C(\mathbb{T})$, let $z \in U(C(\mathbb{T}))$ be the standard unitary generator, one writes that

$$\mathrm{bott}_1(u,\, h) = \mathrm{bott}(u,\, h(z)).$$

Suppose that there is a continuous path of unitaries $u(t) : [0, 1] \to U_0(A)$ such that

$$u(0) = u, \ \ u(1) = 1_A \ \text{ and } \ \|[h(f),\, u(t)]\| < \delta_{C,\mathcal{P},\mathbf{b}} \ \text{ for all } \ t \in [0, 1], \tag{e 2.14}$$

then

$$\mathrm{Bott}(u,\, h)|_{\mathcal{P}} = 0. \tag{e 2.15}$$

Now suppose that C is a unital separable amenable C^*-algebra which is the closure of $\cup_{n=1}^{\infty} C_n$, where $C_n = \lim_{n\to\infty}(C_{n,m}, \varphi_m^{(n)})$ and $K_i(C_{n,m})$ is finitely generated $(i = 0, 1)$. Let z be the standard unitary generator of $C(\mathbb{T})$. We may view $\mathcal{P}$ as a subset of $\underline{K}(C \otimes C(\mathbb{T}))$. Let $\mathcal{G}_0$ be a finite subset of C. Define $\mathcal{G}_1 = \{g \otimes f : g \in \mathcal{G}_0 \ \text{ and } \ f \in \mathcal{S}\}$, where $S = \{1, z, z^*\}$. Let $\mathcal{P}_1 = \mathcal{P} \cup \boldsymbol{\beta}(\mathcal{P})$ (see 2.10 of [17] for the definition of $\boldsymbol{\beta}$). Let $\delta > 0$. Suppose that $(\delta, \mathcal{G}_1, \mathcal{P}_1)$ is a KL-triple for $C \otimes C(\mathbb{T})$ (by selecting large $\mathcal{G}_0$ to begin with).

By choosing even smaller $\delta_{C,\mathcal{P},\mathbf{b}}$, we may assume that, if there is a unitary $u \in A$ such that (e 2.13) holds, and if there is a unital δ-$\mathcal{G}_1$-multiplicative contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ such that

$$\|L(f \otimes 1) - h(f)\| \ < \ \delta_{C,\mathcal{P},\mathbf{b}} \ \text{ for all } \ f \in \mathcal{F}_{C,\mathcal{P},\mathbf{b}} \tag{e 2.16}$$

$$\text{and } \ \|u - L(1 \otimes z)\| \ < \ \delta_{C,\mathcal{P},\mathbf{b}}, \tag{e 2.17}$$

then

$$\mathrm{Bott}(u, h)|_{\mathcal{P}} = [L]|_{\boldsymbol{\beta}(\mathcal{P})}.$$

The following is a restatement of Theorem 7.4 of [17].

Theorem 2.11. *Let X be a compact metric space. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists $\eta > 0$ satisfying the following: For any $\sigma > 0$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following: Suppose that $\varphi : C(X) \to M_n$ is a unital homomorphism such that*

$$\mu_{tr\circ\varphi}(O_r) \geq \sigma$$

for all open ball O_r with radius $r \geq \eta$. If $u \in M_n$ is a unitary such that

$$\|[u, \varphi(g)]\| < \delta \text{ for all } g \in \mathcal{G} \text{ and } \mathrm{Bott}(h, u)|_{\mathcal{P}} = 0,$$

then there exists a continuous rectifiable path of unitaries $\{u_t : t \in [0,1]\}$ of M_n such that

$$u_0 = 1, u_1 = 1_A \text{ and } \|[h(f), u_t]\| < \epsilon$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$. Moreover,

$$\mathrm{Length}(\{u_t\}) \leq 2\pi + \epsilon\pi.$$

3 Almost multiplicative maps from $C(X)$ into interval algebras

Lemma 3.1. *Let X be a compact metric space, let $G \subset K_1(C(X))$ be a finitely generated subgroup generated by $\{s_1, s_2, ..., s_{m(X)}\}$. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset C(X)$ and any finite subset $\mathcal{P} \subset \underline{K}(C(X))$, there exists $\eta > 0$ satisfying the following: For any $1 > \sigma > 0$, there exists $d > 0$, for any $\alpha : KK(C(X) \otimes C(\mathbb{T}), \mathbb{C}) = Hom_\Lambda(\underline{K}(C(X) \otimes C(\mathbb{T}), \underline{K}(\mathbb{C}))$ and for any unital homomorphism $\varphi : C(X) \to M_n$ for some integer $n \geq 1$ for which*

$$\mu_{\tau\circ\varphi}(O_r) \geq \sigma \tag{e 3.18}$$

for any open balls O_r with radius $r \geq \eta$, where tr is the normalized trace on M_n, and

$$\max\{|\alpha(g_i)| : 1 \leq i \leq m(X)\}/n < d, \tag{e 3.19}$$

there exists a unitary $u \in M_n$ such that

$$\|[\varphi(f), u]\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and } \mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \tag{e 3.20}$$

Proof. Let $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. Let $\epsilon_1 = \min\{\epsilon/2, \delta_{C(X),\mathcal{P},\mathbf{b}}\}$ and let $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{F}_{C(X),\mathcal{P},\mathbf{b}}$.

Let $\eta > 0$ be given by 2.8 associated with $\epsilon/16$ (in place of ϵ) and $\mathcal{F}$. Let $\sigma > 0$. Let $\gamma > 0$, $\delta > 0$, $\mathcal{G}$, $\mathcal{P} \subset \underline{K}(C(X))$ and $\mathcal{H} \subset C(X)$ be given by 2.8 associated with the above $\epsilon/16$ (in place of ϵ), $\eta_1 > 0$ (in place of η) and $\sigma/2$. For convenience, we may assume that $\mathcal{H} \cup \mathcal{F} \subset \mathcal{G}$. We may assume that $\delta < \min\{\epsilon/2, 1/4\}$, $\|g\| \leq 1$ if $g \in \mathcal{G}$ and $1_{C(X)} \in \mathcal{G}$.

Let $\mathcal{G}_1 = \{g \otimes f : g \in \mathcal{G} \text{ and } f = 1, z, z^*\} \subset C(X) \otimes C(\mathbb{T})$, where z is the identity function on the unit circle. We may also assume that $(\delta, \mathcal{G}_1, \mathcal{P}_1)$ is a KL-triple for $C(X) \otimes C(\mathbb{T})$. Moreover, we may assume that $\delta < \delta_{C(X),\mathcal{P},\mathbf{b}}$ and $\mathcal{G} \supset \mathcal{F}_{C(X),\mathbf{b}}$.

Suppose that $C(X) = \lim_{n\to\infty} C(Y_n)$, where each Y_n is a finite CW complex. Let $\imath_m : C(Y_m) \to C(X)$ be the unital homomorphism induced by the inductive limit system. We may assume that there is a finite subset $\mathcal{G}' \subset C(Y_m)$ and there is a finite subset $\mathcal{P}' \subset \underline{K}(C(Y_m))$ such that $\imath_m(\mathcal{G}') \supset \mathcal{G}$ and $[\imath_m](\mathcal{P}') \supset \mathcal{P}$. We may also assume that there are $s'_1, s'_2, ..., s'_{m(X)} \in$

$K_1(C(Y_m))$ such that $(\imath_m)_{*1}(s'_j) = s_j$, $j = 1, 2, ..., m(X)$. Let $\mathcal{G}'_1 = \{g \otimes f : g \in \mathcal{G}' \text{ and } f = 1, z, z^*\}$. We may further assume that $(\delta, \mathcal{G}')$ is a KK-pair for $C(Y_m) \otimes C(\mathbb{T})$.

Suppose that Y_m is the disjoint union of finitely many connected CW complexes $Z_1, Z_2, ..., Z_l$. Without loss of generality, we may assume that there is, for each i, a finite subset $\mathcal{G}^{(i)} \subset C(Z_i)$ such that $\oplus_{i=1}^l \mathcal{G}^{(i)} = \mathcal{G}'$ and there is a finite subset $\mathcal{P}'_i \subset \underline{K}(C(Z_i))$ such that $\oplus_{i=1}^l \mathcal{P}'_i = \mathcal{P}'$. Choose $\xi_i \in Z_i$ such that $\xi_i \in Y$, $i = 1, 2, ..., l$.

Let $N(\delta/4, \mathcal{G}_1^{(i)}, \mathcal{P}'_i)$ be given by Lemma 10.2 of [18] for $C(\mathbb{T} \times Z_i)$. Define

$$N(\delta/4, \mathcal{G}'_1, \mathcal{P}') = \sum_{i=1}^{l} N(\delta/4, \mathcal{G}_1^{(i)}, \mathcal{P}'_i).$$

Let

$$d = \min\{\sigma/2, \gamma\} \cdot \frac{1}{N(\delta/4, \mathcal{G}'_1, \mathcal{P}')}.$$

Let α be as in the statement and let

$$k = \max\{|\alpha(s_i)| : i = 1, 2, ..., m(X)\}.$$

Note that, if $x \in \ker\rho_{C(Y_m)}$, then $h_{*0}(x) = 0$. Let Y be the compact subset of Y_m such that $\imath_m(C(Y_m)) = C(Y)$. Denote by $\imath : C(Y) \to C(X)$ the embedding given by $\imath_m$. Let $s : X \to Y$ be the surjective map such that $\imath(f)(x) = f(s(x))$ for all $f \in C(Y_m)$ and $x \in X$.

Choose $\beta \in Hom_\Lambda(\underline{K}(C(Y_m) \otimes C(\mathbb{T})), \underline{K}(\mathbb{C}))$ defined as

$$\beta|_{\underline{K}(C(Y_m))} = [H] \tag{e 3.21}$$

for some point-evaluation (at $\xi_1, \xi_2, ..., \xi_l$) $H : C(Y_m) \to M_K$ (for some integer $K \geq 1$) and

$$\beta|_{\boldsymbol{\beta}(\underline{K}(C(Y_m)))} = \alpha \circ [\imath_m]|_{\boldsymbol{\beta}(\underline{K}(C(Y_m)))} \tag{e 3.22}$$

(see 2.10 of [17] for the definition of $\boldsymbol{\beta}$). Let $\mathcal{G}_1 = \{g \otimes 1 : g \in \mathcal{G}\} \cup \{z_1\} \subset C(Y_m) \otimes C(\mathbb{T})$, where $z_1 = 1 \otimes z$ and z is the identity function on the unit circle. Let $L = kN(\delta, \mathcal{G}'_1, \mathcal{P}')$.

It follows from Lemmas 10.2 of [18] that there exists a unital $\delta/4$-$\mathcal{G}'_1$-multiplicative contractive completely positive linear map $\Phi : C(Y_m) \otimes C(\mathbb{T}) \to M_L$ such that

$$[\Phi]|_{\underline{K}(C_0(Z))} = \beta|_{\underline{K}(C_0(Z))}, \tag{e 3.23}$$

where $Z = Y \times \mathbb{T} \setminus \cup_{i=1}^l \{\xi_i \times 1_{C(\mathbb{T})}\}$ and where 1 is the point in the unit circle. Define $\varphi'_0 : C(X) \to M_L$ by $\varphi'_0(f) = \Phi(f \otimes 1_{C(\mathbb{T})})$ for all $f \in C(X)$. Define

$$u_0 = L(1_{C(X)} \otimes z)(L(1_{C(X)} \otimes z^*)L(1_{C(X)} \otimes z))^{1/2} = \langle L(1_{C(X)} \otimes z)\rangle.$$

Then

$$\|u_0 - L(1_{C(X)} \otimes z)\| < \delta_{C(X),\mathcal{P},\mathbf{b}}.$$

Now suppose that $\varphi : C(X) \to M_n$ for some integer $n \geq 1$ for which

$$\mu_{tr\circ\varphi}(O_r) \geq \sigma \tag{e 3.24}$$

for all open balls O_r with radius $r \geq \eta$, where tr is the normalized tracial state on M_n and

$$k/n < d.$$

Note that $n \ge L$. We may write that

$$\varphi(f) = \sum_{i=1}^{n} f(\xi_i)p_i \ \text{ for all } \ f \in C(X), \tag{e 3.25}$$

where $\{p_1, p_2, ..., p_n\}$ is a set of mutually orthogonal rank one projections and $\xi_i \in X$, $i = 1, 2, ..., n$. Define $\varphi' : C(X) \to M_{n-L}$ defined by

$$\varphi'(f) = \sum_{i=1}^{n-L} f(\xi_i)p_i \ \text{ for all } \ f \in C(X). \tag{e 3.26}$$

Define $\varphi_1 : C(X) \to M_n$ by

$$\varphi_1(f) = \varphi'(f) \oplus \varphi_0(f) \ \text{ for all } \ f \in C(X). \tag{e 3.27}$$

Since $k/n < d \le \gamma(k/L)$, $L/n < \gamma$. Therefore one computes that

$$|\tau \circ \varphi(g) - \tau \circ \varphi_1(f)| < \gamma \ \text{ for all } \ g \in \mathcal{H}. \tag{e 3.28}$$

Moreover, since $k/n < d \le (\sigma/2)k/L$, $L/n < \sigma/2$. Therefore, by (e 3.24),

$$\mu_{tr\circ\varphi_1}(O_r) \ge \sigma/2$$

for all $r \ge \eta$.

It follows from 2.8 (also using (e 3.21)) that there is a unitary $w \in M_n$ such that

$$\|\mathrm{Ad}\, w \circ \varphi_1(f) - \varphi(f)\| < \epsilon/2 \ \text{ for all } \ f \in \mathcal{F}. \tag{e 3.29}$$

Put

$$u = w^*(\mathrm{diag}(\overbrace{1, 1, ..., 1}^{n-L}, u_0))w. \tag{e 3.30}$$

One check that this unitary u meets all the requirements.

□

The following is a folklore.

Lemma 3.2. *Let X be a compact metric space, let $\eta_i > 0$ and $\sigma_i > 0$ $(i = 1, 2, ..., m)$ with $\eta_1 > \eta_2 > \cdots > \eta_m$ and $\sigma_1 > \sigma_2 > \cdots > \sigma_m$, and let $0 < \lambda_1, \lambda_2 < 1$. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following:*

Suppose that A is a unital C^*-algebra with $T(A) \ne \emptyset$ and suppose that $\varphi, \psi : C(X) \to A$ are two unital positive linear maps such that

$$\mu_{\tau\circ\varphi}(O_r) \ge \sigma_j \tag{e 3.31}$$

for all $r \ge \eta_j$, $j = 1, 2, ..., m$, and

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \delta \ \text{ for all } \ g \in \mathcal{G}. \tag{e 3.32}$$

Then,

$$\mu_{\tau\circ\psi}(O_r) \ge \lambda_1\sigma_j$$

for all $r \ge 2(1 + \lambda_2)\eta_j$, $j = 1, 2, ..., m$.

Proof. To simplify the proof, without loss of generality, we will prove only for the case that $m = 1$. The general case follows by taking minimum of m δ's and the union of m $\mathcal{G}'s$.

There are $x_1, x_2, ..., x_K \in X$ such that

$$\cup_{k=1}^{K} O(x_k, \eta) \supset X.$$

There are $f_1, f_2, ..., f_K \in C(X)$ with $0 < f_k \le 1$ such that $f_k(x) = 1$ if $x \in O(x_k, \eta)$ and $f_k(x) = 0$ if $\mathrm{dist}(x, x_k) > (1 + \lambda_2)\eta$. Choose $\delta = (1 - \lambda_1)\sigma_1$ and $\mathcal{G} = \{f_1, f_2, ..., f_K\}$.

Now suppose that $\varphi, \psi : C(X) \to A$ are two unital positive linear maps which satisfy the assumption (e 3.31) and (e 3.32).

Let $x \in X$ and consider $O(x, r)$ for some $r \ge 2(1 + \lambda_2)\eta$. Then there exists x_k such that $\mathrm{dist}(x, x_k) < \eta$. This implies that

$$O(x_k, (1 + \lambda_2)\eta) \subset O(x, r).$$

Thus

$$\begin{aligned} \mu_{\tau\circ\psi}(O(x, r)) &\ge \tau \circ \psi(f_k) > \tau \circ \varphi(f_k) - (1 - \lambda_1)\sigma_1 && \text{(e 3.33)} \\ &\ge \mu_{\tau\circ\varphi}(O(x_k, \eta)) - (1 - \lambda_1)\sigma_1 && \text{(e 3.34)} \\ &\ge \lambda_1\sigma_1. && \text{(e 3.35)} \end{aligned}$$

for all $\tau \in T(A)$.

□

Remark 3.3. Note that in the above lemma, we insist that δ and $\mathcal{G}$ do not depend on φ. Otherwise one can have better estimates.

Lemma 3.4. *Let X be a compact metric space, let $\Delta : (0, 1) \to (0, 1)$ be a nondecreasing function, let $\eta > 0$ and let $0 < \lambda_1, \lambda_2 < 1$. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following:*

Suppose that A is a unital C^*-algebra with $T(A) \ne \emptyset$ and suppose that $\varphi, \psi : C(X) \to A$ are two unital positive linear maps such that

$$\mu_{\tau\circ\varphi}(O_r) \ge \Delta(r) \tag{e 3.36}$$

for all $r \ge \eta$ and

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}. \tag{e 3.37}$$

Then,

$$\mu_{\tau\circ\psi}(O_r) \ge \lambda_1\Delta(r/2(1 + \lambda_2))$$

for all $r \ge 2(1 + \lambda_2)\eta$.

Proof. Let $\eta > 0$, Δ and $0 < \lambda_1, \lambda_2 < 1$ be given. Choose $\lambda_0 > 0$ such that $0 < \lambda_0 < \lambda_2$. Let $1 > r_1 > r_2 > \cdots > r_N > 0$ such that $\eta > r_N$ and

$$r_{i+1}/r_i > \frac{1 + \lambda_0}{1 + \lambda_2}, \quad i = 1, 2, ..., N - 1.$$

Put $\eta_j = r_j$ and $\sigma_j = \Delta(\eta_j)$, $j = 1, 2, ..., N - 1$.

Let $\delta > 0$ and $\mathcal{G}$ be required by 3.2 for η_j and σ_j $(j = 1, 2, ..., N)$,λ_1 and λ_2.

Now suppose that φ, ψ satisfy (e 3.36) and (e 3.37). By applying 3.2, we conclude that

$$\mu_{\tau\circ\psi}(O_r) \ge \lambda_1\sigma_j$$

for all $\tau \in T(A)$ and all $r \ge 2(1+\lambda_0)\eta_j$, $j = 1, 2, ..., N$.

Now suppose that $r \ge 2(1+\lambda_2)\eta > 2(1+\lambda_0)\eta$. Then

$$\frac{r}{2(1+\lambda_0)} > \eta.$$

We may assume that, for some j,

$$\eta_j > \frac{r}{2(1+\lambda_0)} > \eta_{j+1}.$$

Then

$$\begin{aligned} \mu_{\tau\circ\psi}(O_r) &> \lambda_1\sigma_{j+1} = \lambda_1\Delta(\eta_{j+1}) && (\text{e } 3.38)\\ &\ge \lambda_1\Delta(\eta_j(\frac{1+\lambda_0}{1+\lambda_2})) && (\text{e } 3.39)\\ &\ge \lambda_1\Delta(\frac{r}{2(1+\lambda_2)}) && (\text{e } 3.40) \end{aligned}$$

for all $\tau \in T(A)$.

□

Lemma 3.5. *Let $u \in CU(M_n(C([0,1]))$ be a unitary such that*

$$\|u(0)u(t)^* - 1\| < 1 \text{ for all } t \in [0,1]. \qquad (\text{e } 3.41)$$

Suppose that $u(0)u(1)^* = \exp(\sqrt{-1}h)$ with $\|h\| < 2\arcsin(1/2)$. Then

$$Tr(h) = 0.$$

Proof. Write $u = \exp(\sqrt{-1}a)$, where $a \in M_n(C([0,1])$ is a selfadjoint element. It follows that

$$(\frac{1}{2\pi})Tr(a(t)) \in \mathbb{Z}.$$

Therefore $Tr(a(t))$ is a constant. There exists a selfadjoint element $b \in M_n(C([0,1])$ such that

$$u(0)u(t)^* = \exp(\sqrt{-1}b(t)) \text{ and } \|b\| < 2\arcsin(1/2).$$

However, $u(0)u(t)^* \in CU(M_n(C([0,1]))$. Thus, from what have been proved above, $(\frac{1}{2\pi})Tr(b(t))$ is a constant. Since $b(0) = 0$,

$$(\frac{1}{2\pi})Tr(b(t)) = 0 \text{ for all } t \in [0,1].$$

Note that $h = b(1)$. Therefore

$$Tr(h) = 0.$$

□

Theorem 3.6. *Let X be a compact metric space, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\epsilon > 0$ be a positive number. There exists $\eta_1 > 0$ satisfying the following: for any $\sigma_1 > 0$, there exists $\eta_2 > 0$ satisfying the following: for any $\sigma_2 > 0$, there exists $\eta_3 > 0$ satisfying the following: for any $\sigma_3 > 0$, there exists $\eta_4 > 0$ satisfying the following: For any $\sigma_4 > 0$, there exists $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ a finite subset $\mathcal{H} \subset C(X)$ and a finite subset $\mathcal{U} \subset U_c(K_1(C(X)))$ for which $[\mathcal{U}] \subset \mathcal{P}$ satisfying*

the following: For any two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $\varphi, \psi : C(X) \to M_n(C([0,1])$ such that

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} = [h]|_{\mathcal{P}} \tag{e 3.42}$$

for some unital homomorphism $h : C(X) \to M_n(C([0,1]))$,

$$\mu_{\tau\circ\varphi}(O_r) \ge \sigma_i, \ \ \mu_{\tau\circ\psi}(O_r) \ge \sigma_i, \tag{e 3.43}$$

for all $\tau \in T(M_n(C([0,1])))$ and for all $r \ge \eta_i$, $i = 1, 2, 3$,

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \gamma_1 \ \text{for all } g \in \mathcal{H} \ \text{and} \tag{e 3.44}$$

$$\mathrm{dist}(\overline{\langle \varphi(u)\rangle}, \overline{\langle \psi(u)\rangle}) < \gamma_2 \ \text{for all } u \in \mathcal{U}, \tag{e 3.45}$$

there exists a unitary $W \in M_n(C([0,1]))$ such that

$$\|W\varphi(f)W^* - \psi(f)\| < \epsilon \ \text{for all } f \in \mathcal{F}. \tag{e 3.46}$$

(Note, as stated in 2.1, φ and ψ in (e 3.44) is in fact $\varphi \otimes \mathrm{id}_{M_k}$ and $\psi \otimes \mathrm{id}_{m_k}$ for some integer $k \ge 1$. This will be used in the proof below.)

Proof. Put $B = M_n(C([0,1])$.

We may write $C(X) = \lim_{n\to\infty}(C(Y_n), \imath_n)$, where Y_n is a finite CW complex. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given. Without loss of generality, we may assume that $\mathcal{F} \subset \imath_n(C(Y_n))$ for some n. Let $\eta_1' > 0$ (in place of η) be required by 2.11 for $\epsilon/32$ (in place of ϵ) and $\mathcal{F}$.

Let $\eta_1 = \eta_1'/3$. Let $\sigma_1 > 0$ and let $\sigma_1' = \sigma_1/2 > 0$. Let $\delta_1 > 0$ (in place of ϵ), $\mathcal{G}_1 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset and let $\mathcal{P}_0 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) be a finite subset required by 2.11 for $\epsilon/32$ (in place of ϵ), $\mathcal{F}$, η_1' and σ_1'. We may assume that $\delta_1 < \epsilon/32$.

There exists a finite CW complex Y, a unital homomorphism $\imath : C(Y) \to C(X)$ and a finite subset $\mathcal{F}' \subset C(Y)$ such that $\imath(\mathcal{F}') = \mathcal{F}$ and $[\imath](\underline{K}(C(Y))) \supset \mathcal{P}_0$ (by choosing $Y = Y_n$ for some large n).

Let $0 < \delta_2 < \delta_{C(Y),\mathbf{b}}$ and $\mathcal{G}_2' \supset \mathcal{F}_{C(Y),\mathbf{b}}$ such that $(\delta_2, \mathcal{G}_2')$ forms a KK-pair for $C(Y)$. Let $\mathcal{P}_0' \subset \underline{K}(C(Y))$ be such that $\delta_{C(Y),\mathbf{b}} = \delta_{C,\mathcal{P}_0',\mathbf{b}}$. To simplify the notation, without loss of generality, we may assume that $[\imath](\mathcal{P}_0') = \mathcal{P}_0$. Put $\mathcal{G}_2 = \imath(\mathcal{G}_2')$.

Denote by $z \in C(\mathbb{T})$ the identity function on the unit circle. We may also assume that, for any δ_2-$\{z, 1\} \times \mathcal{G}_2$-multiplicative contractive completely positive linear map $\Lambda : C(\mathbb{T}) \otimes C(Y) \to C$ (for any unital C^*-algebra C with $T(C) \ne \emptyset$), $[\Lambda]$ is well defined and

$$\tau([\Lambda(g)]) = 0$$

for all $g \in Tor(K_1(C(Y)))$ (which is a finite subgroup).

Furthermore, we may assume that δ_2 is so small that if $\|uv - vu\| < 3\delta_2$, then the Exel formula

$$\tau(\mathrm{bott}_1(u, v)) = \frac{1}{2\pi\sqrt{-1}}(\tau(\log(u^*vuv^*))$$

holds in any unital C^*-algebra C with tracial rank zero and any $\tau \in T(C)$ (see Theorem 3.6 of [16]). Moreover if $\|v_1 - v_2\| < 3\delta_2$, then

$$\mathrm{bott}_1(u, v_1) = \mathrm{bott}_1(u, v_2).$$

Let $\mathcal{U} = \{g_1, g_2, ..., g_{k(X)}\} \subset U_c(K_1(C(X))$ be a finite subset such that $\{[g_1], [g_2], ..., [g_{k(X)}]\}$ forms a set of generators for the finitely generated subgroup generated by $\mathcal{P}_0 \cap K_1(C(X))$. We assume that $m(X) \ge 1$ is an integer and $g_i \in U(M_{m(X)}(C(X))$. We may further assume that

there are g_j' ($j=1,2,...,k(X)$) in $U_c(K_1(C(Y)))$ such that $\imath(g_j)=g_j'$, $j=1,2,...,k(X)$ (here again we identify a set of unitaries with its image in $U(C(Y))/CU(C(Y))$). Furthermore, we may assume that $g_1', g_2', ..., g_{k(X)}'$ generate $K_1(C(Y))$. Let $\mathcal{U}_0 \subset C(X)$ be a finite subset such that

$$\mathcal{U} = \{(a_{i,j}) : a_{i,j} \in \mathcal{U}_0\}.$$

Let $\delta_u = \min\{1/256m(X)^2, \delta_1/16m(X)^2, \delta_2/16m(X)^2\}$ and $\mathcal{G}_u = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{U}_0$.

Let $\eta_2' > 0$ (in place of η) required by 3.1 for δ_u (in place of ϵ) and $\mathcal{G}_u$ (in place of $\mathcal{F}$). Put $\eta_2 = \eta_2'/3$.

Let $\sigma_2 > 0$ and let $\sigma_2' = \sigma_2/2$. Let $1 > d > 0$ be required by 3.1 for $\min\{\delta_1/4, \delta_2/4\}$ (in place ϵ), $\mathcal{G}_u$ (in place of $\mathcal{F}$), η_2 and σ_2'.

Let $\delta_3 > 0$ (in place of δ) and let $\mathcal{G}_3 \subset C(\mathbb{T}) \otimes C(X)$ (in place of $\mathcal{G}$) required by Lemma 10.3 of [18] for $d/8$ (in place of σ) and $\mathbb{T} \times X$ (in place of X). Without loss of generality, we may assume that

$$\mathcal{G}_3 = \{z \otimes g : g \in \mathcal{G}_4'\} \cup \{1 \otimes g : g \in \mathcal{G}_4\},$$

where $\mathcal{G}_4' \subset C(X)$ is a finite subset (by choosing a smaller δ_3 and large $\mathcal{G}_3$).

Let $\epsilon_1' > 0$ (in place of δ) and let $\mathcal{G}_4'' \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset required by 3.2 for η_1, η_2, σ_1, σ_2, $1/2$ (in place of λ_1) and $1/4$ (in place of λ_2).

Let $\epsilon_1'' = \min\{d/27m(X)^2, \delta_u/2, \delta_3/2m(X)^2, \epsilon_1'/2m(X)^2\}$ and let $\bar{\epsilon}_1 > 0$ (in place of δ) and $\mathcal{G}_5 \subset C(X)$ (in place of $\mathcal{F}_1$) be a finite subset required by 2.8 of [17] for ϵ_1'' (in place of ϵ) and $\mathcal{G}_u \cup \mathcal{G}_4' \cup \mathcal{G}_4''$ (and $C(X)$ in place of B). Put

$$\epsilon_1 = \min\{\epsilon_1', \epsilon_1'', \bar{\epsilon}_1\}.$$

Let $\eta_3' > 0$ (in place of η) be required by 2.8 for $\epsilon_1/4$ (in place ϵ) and $\mathcal{G}_5$ (in place of $\mathcal{F}$).

Let $\eta_3'' > 0$ (in place of η_1) be required by 2.9 for $\epsilon_1/4$ (in place of ϵ) and $\mathcal{G}_5$ (in place of $\mathcal{F}$). Let $\eta_3 = \min\{\eta_3', \eta_3''\}$. Let $\sigma_3 > 0$. Let $\gamma_1 > 0$ (in place of γ), $\delta_4 > 0$, $\mathcal{G}_6 \subset C(X)$ (in place of $\mathcal{G}$), $\mathcal{H} \subset C(X)$ be a finite subset and let $\mathcal{P}_1 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) be required by 2.8 for $\epsilon_1/4$ (in place of ϵ), $\mathcal{G}_5$ (in place of $\mathcal{F}$), η_3 (in place η) and σ_3 (in place σ). Let $\eta_4 > 0$ (in place of η_2) be required by 2.9 for $\epsilon_1/4$ (in place of ϵ), $\mathcal{G}_5$ (in place of $\mathcal{F}$), η_3 (in place of η_1), σ_3 (in place of σ_1). Let $\sigma_4 > 0$. Let $\delta_5 > 0$, $\mathcal{G}_7 \subset C(X)$ (in place of $\mathcal{G}$),$\mathcal{P}_2 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) required by 2.9.

Let $\delta = \min\{\epsilon_1/4, \delta_4, \delta_5\}$, $\mathcal{G} = \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{H}$ and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2$. Let $\gamma_2 < \min\{d/16m(X)^2, \delta_u/9m(X)^2, 1/256m(X)^2\}$. We may assume that $(\delta, \mathcal{G}, \mathcal{P})$ is a KL-triple.

Now suppose that $\varphi, \psi : C(X) \to B$ for some integer n are two unital δ-$\mathcal{G}$ multiplicative contractive completely positive linear maps which satisfies the assumption for the above η_i, δ_i ($i=1,2,3,4$), γ_i ($i=1,2$), $\mathcal{P}$, $\mathcal{U}$ and $\mathcal{H}$.

Choose a partition

$$0 = t_0 < t_1 < \cdots t_N = 1$$

such that

$$\|\pi_t \circ \varphi(g) - \pi_{t_{i-1}} \circ \varphi(g)\| < \epsilon_1/4 \text{ and } \|\pi_t \circ \psi(g) - \pi_{t_{i-1}} \circ \psi(g)\| < \epsilon_1/4 \tag{e 3.47}$$

for all $g \in \mathcal{G}$ and for all $t \in [t_{i-1}, t_i]$, $i = 1, 2, ..., N$. By applying 2.8, for each i, there exists a unitary $w_i \in M_n$ such that

$$\|w_i \pi_{t_i} \circ \varphi(g) w_i^* - \pi_{t_i} \circ \psi(g)\| < \epsilon_1/4 \text{ for all } g \in \mathcal{G}_5 \tag{e 3.48}$$

and, by 2.9, there are unital homomorphisms $h_{i,1}, h_{i,2} : C(X) \to M_n$ such that

$$\|\pi_{t_i} \circ \varphi(g) - h_{i,1}(g)\| < \epsilon_1/4 \text{ and } \|\pi_{t_i} \circ \psi(g) - h_{i,2}(g)\| < \epsilon_1/4 \text{ for all } g \in \mathcal{G}_5, \tag{e 3.49}$$

$i = 0, 1, 2, ..., N$. Moreover (by also applying 3.2),

$$\mu_{tr\circ h_{i,j}}(O_r) \ge \sigma'_k \tag{e 3.50}$$

for all $r \ge \eta'_k$, $k = 1, 2$, $j = 1, 2$ and $i = 1, 2, ..., N$. Let $\omega_j \in M_{m(X)}(B)$ be a unitary such that $\omega_j \in CU(M_{m(X)}(B)$ and

$$\|\langle\varphi(g_j^*)\rangle\langle\psi(g_j)\rangle - \omega_j\| < \gamma_2, \ \ i = 1, 2, ..., k(X). \tag{e 3.51}$$

Write

$$\omega_j = \exp(\sqrt{-1}a_j)$$

for some selfadjoint element $a_j \in M_{m(X)}(M_n(C([0, 1])))$, $j = 1, 2, ..., k(X)$. Then

$$\frac{n(t \otimes Tr_{m(X)})(a_j(s))}{2\pi} \in \mathbb{Z}$$

($s \in [0, 1]$), where t is the normalized trace on M_n. It follows that the above is a constant. In particular,

$$n(t \otimes Tr_{m(X)})(a_j(t_i)) = n(t \otimes Tr_{m(X)})(a_j(t_{i-1})), \tag{e 3.52}$$

$i = 1, 2, ..., N$ and $j = 1, 2, ..., m(X)$.

Let $W_i = w_i \otimes \mathrm{id}_{M_{m(X)}}$, $i = 0, 1, ..., N$. Then

$$\|(h_{i,1} \otimes \mathrm{id}_{M_{m(X)}})(g_j^*)W_i(h_{i,1} \otimes \mathrm{id}_{M_{m(X)}})(g_j)W_i^* - \omega_j(t_i)\| < 3m(X)^2\epsilon_1 + 2\gamma_2 < 1/32 \tag{e 3.53}$$

It follows from (e 3.51) that there exists selfadjoint elements $b_{i,j} \in M_{nm(X)}$ such that

$$\exp(\sqrt{-1}b_{i,j}) = \omega_j(t_i)^*(h_{i,1} \otimes \mathrm{id}_{M_{m(X)}})(g_j^*)W_i(h_{i,1} \otimes \mathrm{id}_{M_{m(X)}})(g_j)W_i^* \tag{e 3.54}$$

such that

$$\|b_{i,j}\| < 2\arcsin(3m(X)^2\epsilon_1/4 + 2\gamma_2), \ \ j = 1, 2, ..., m(X), \ i = 0, 1, ..., N. \tag{e 3.55}$$

Note that

$$(h_{i,1} \otimes \mathrm{id}_{M_{m(X)}})(g_j^*)W_i(h_{i,1} \otimes \mathrm{id}_{M_{m(X)}})(g_j)W_i^* = \omega_j(t_i)\exp(\sqrt{-1}b_{i,j}), \tag{e 3.56}$$

$j = 1, 2, ..., m(X)$ and $i = 0, 1, ..., N$.

Then,

$$\frac{n}{2\pi}(t \otimes Tr_{M_{m(X)}})(b_{i,j}) \in \mathbb{Z}, , \ \ j = 1, 2, ..., m(X), \ i = 0, 1, ..., N. \tag{e 3.57}$$

Let

$$\lambda_{i,j} = \frac{n}{2\pi}(t \otimes Tr_{M_{m(X)}})(b_{i,j})$$

$j = 0, 1, 2, ..., m(X)$, $i = 1, 2, ..., N$. Note that $\lambda_{i,j} \in \mathbb{Z}$.

Define $\alpha_i^{(0,1)} : K_1(C(Y)) \to \mathbb{Z}$ by mapping g'_j to $\lambda_{i,j}$, $j = 1, 2, ..., m(X)$ and $i = 0, 1, 2, ..., N$. We write $K_0(C(\mathbb{T}) \otimes C(Y)) = K_0(C(Y)) \oplus \boldsymbol{\beta}(K_1(C(Y)))$ (see 2.10 of [17] for the definition of $\boldsymbol{\beta}$). Define $\alpha_i : K_*(C(\mathbb{T}) \otimes C(Y)) \to K_*(M_n)$ as follows

$$\alpha_i|_{K_0(C(\mathbb{T})\otimes C(Y))}([1]) = n, \tag{e 3.58}$$

$$\alpha_i|_{\ker\rho_{C(Y)}} = 0, \tag{e 3.59}$$

$$\alpha_i|_{\boldsymbol{\beta}(K_1(C(Y)))} = \alpha_i \circ \boldsymbol{\beta}|_{K_1(C(Y))} = \alpha_i^{(0,1)}, \tag{e 3.60}$$

$$\alpha_i|_{K_1(C(\mathbb{T})\otimes C(Y))} = 0, \tag{e 3.61}$$

$$\tag{e 3.62}$$

By the Universal Coefficient Theorem ([23]), there exists an element $\alpha_i \in KK(C(\mathbb{T})\otimes C(Y),\mathbb{C})$ such that $\alpha_i|_{K_*(C(\mathbb{T})\otimes C(Y)} = \alpha_i$ as defined above, $i = 1, 2, ..., N$. We estimate that

$$\|(w_{i-1}^* w_i)h_{i-1,1}(g) - h_{i-1,1}(g)(w_{i-1}^* w_i)\| < \epsilon_1 \text{ for all } g \in \mathcal{G}_5.$$

Let $\Lambda_i : C(\mathbb{T})\otimes C(X) \to M_n$ be a unital contractive completely positive linear map given by the pair $w_{i-1}^* w_i$ and $h_{i-1,1}$, $i = 1, 2, ..., N$ (see 2.8 of [17]). Denote $V_{i,j} = h_{i,1}\otimes \mathrm{id}_{M_{m(X)}}(g_j)$, $j = 1, 2, ..., m(X)$ and $i = 0, 1, 2, ..., N$. Note that

$$\|W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} V_{i-1,j}^* W_i V_{i-1,j} W_i^* - 1\| < 1/16 \tag{e 3.63}$$

$$\|W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} V_{i,j}^* W_i V_{i,j} W_i^* - 1\| < 1/16 \tag{e 3.64}$$

and there is a continuous path $Z(t)$ of unitaries such that $Z(0) = V_{i-1,j}$ and $Z(1) = V_{i,j}$. We obtain a continuous path

$$W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} Z(t)^* W_i Z(t) W_i^*$$

which is in $CU(M_{nm(X)})$ for all $t \in [0, 1]$. It follows that

$$(1/2\pi\sqrt{-1})(t\otimes Tr_{M_{m(X)}})[\log(W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} Z(t)^* W_i Z(t) W_i^*)]$$

is a constant. In particular,

$$(1/2\pi\sqrt{-1})(t\otimes Tr_{M_{m(X)}})\log(W_{i-1}V_{i-1,j}^* W_{i-1}^* W_i V_{i-1,j} W_i^*) \tag{e 3.65}$$

$$= (1/2\pi\sqrt{-1})(t\otimes Tr_{M_{m(X)}})\log(W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} V_{i,j}^* W_i V_{i,j} W_i^*). \tag{e 3.66}$$

Also

$$W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} V_{i,j}^* W_i V_{i,j} W_i^* \tag{e 3.67}$$

$$= (\omega_j(t_{i-1})\exp(\sqrt{-1}b_{i-1,j}))^* \omega_j(t_i)\exp(\sqrt{-1}b_{i,j}) \tag{e 3.68}$$

$$= \exp(-\sqrt{-1}b_{i-1,j})\omega_j(t_{i-1})^*\omega_j(t_i)\exp(\sqrt{-1}b_{ij}). \tag{e 3.69}$$

Note that, by (e 3.51) and (e 3.53),

$$\|\omega_j(t_{i-1})^*\omega_j(t_i) - 1\| < 3(3\epsilon_1' + 2\gamma_2) < 3/32, \tag{e 3.70}$$

$j = 1, 2, ..., m(X)$, $i = 1, 2, ..., N$. By 3.5

$$(t\otimes Tr_{m(X)})(\log(\omega_j(t_{i-1})^*\omega_j(t_i))) = 0 \tag{e 3.71}$$

It follows that (by the Exel formula, using (e 3.66), (e 3.69) and (e 3.71))

$$(t\otimes Tr_{m(X)})(\mathrm{bott}(V_{i-1,j}, W_{i-1}^* W_i)) \tag{e 3.72}$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t\otimes Tr_{m(X)})(\log(V_{i-1,j}^* W_{i-1}^* W_i V_{i-1,j} W_i^* W_{i-1})) \tag{e 3.73}$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t\otimes Tr_{m(X)})(\log(W_{i-1}V_{i-1,j}^* W_{i-1}^* W_i V_{i-1,j} W_i^*)) \tag{e 3.74}$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t\otimes Tr_{m(X)})(\log(W_{i-1}V_{i-1,j}^* W_{i-1}^* V_{i-1,j} V_{i,j}^* W_i V_{i,j} W_i^*)) \tag{e 3.75}$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t\otimes Tr_{m(X)})(\log(\exp(-\sqrt{-1}b_{i-1,j})\omega_j(t_{i-1})^*\omega_j(t_i)\exp(\sqrt{-1}b_{ij})) \tag{e 3.76}$$

$$= (\frac{1}{2\pi\sqrt{-1}})[(t\otimes Tr_{k(n)})(-\sqrt{-1}b_{i-1,j}) + (t\otimes Tr_{k(n)})(\log(\omega_j(t_{i-1})^*\omega_j(t_i)) \tag{e 3.77}$$

$$+(t\otimes Tr_{k(n)})(\sqrt{-1}b_{i,j})] \tag{e 3.78}$$

$$= \frac{1}{2\pi}[(t\otimes Tr_{k(n)})(-b_{i-1,j} + b_{i,j}). \tag{e 3.79}$$

In other words,

$$\mathrm{bott}(V_{i-1,j}, W_{i-1}^* W_i)) = -\lambda_{i-1,j} + \lambda_{i,j} \tag{e 3.80}$$

$j = 1, 2, ..., m(X)$ and $i = 1, 2, ..., N$.

Define $\beta_0 = 0$, $\beta_1 = [\Lambda_1] - \alpha_0 + \alpha_1$ and

$$\beta_i = [\Lambda_i] - \alpha_{i-1} + \alpha_i, \ \ i = 1, 2, ..., N. \tag{e 3.81}$$

Define $\kappa_0 = 0$ and $\kappa_i = \alpha_i + \beta_i$, $i = 1, 2, ..., N$. Note that $\alpha_i, \beta_i, \kappa_i \in KK(C(\mathbb{T}) \otimes C(Y)), \mathbb{C})$ We compute that

$$\begin{aligned}
\beta_1(g_j') &= [\Lambda_1](g_j') - \lambda_{0,j} + \lambda_{1,j} = 2(\lambda_{1,j} - \lambda_{0,j}), && \text{(e 3.82)}\\
\beta_2(g_j') &= [\Lambda_2](g_j') - \lambda_{1,j} - \lambda_{2,j} + \beta_1(g_j') && \text{(e 3.83)}\\
&= 2(\lambda_{2,j} - \lambda_{0,j}) \text{ and} && \text{(e 3.84)}\\
\beta_i(g_j') &= 2(\lambda_{i,j} - \lambda_{0,j}), \ \ i = 1, 2, ..., N \text{ and } \ \ j = 1, 2, ..., k(X). && \text{(e 3.85)}
\end{aligned}$$

It follows that

$$|\tau \otimes Tr_{m(X)}(\kappa_i([g_i]))| \ \ = \ \ 4|\lambda_{i,j}/n| + 2|\lambda_{0,j}| < d/2, \tag{e 3.86}$$

$j = 1, 2, ..., N$ and $i = 1, 2, ..., k(X)$. By applying 3.1, there is, for each $i = 1, 2, ..., N$, a unitary $z_i \in M_n$ such that

$$\|[z_i, \ h_{i,1}(g)]\| < \delta_u \ \text{ for all } \ g \in \mathcal{G}_u \tag{e 3.87}$$

$$\text{and } \ \mathrm{Bott}(z_i, h_{i,1} \circ \imath) = \kappa_i, \ \ i = 0, 1, 2, ..., N-1. \tag{e 3.88}$$

Let $U_i = z_{i-1} w_{i-1}^* w_i z_i^*$, $i = 1, 2, ..., N$. Then

$$\|[U_i, h_{i-1,1}(g)]\| < \min\{\delta_1, \delta_2\} \ \text{ for all } \ g \in \mathcal{G}_u, \tag{e 3.89}$$

$i = 1, 2, ..., N$. Moreover

$$\begin{aligned}
\mathrm{Bott}(U_i, h_{i-1,1} \circ \imath) &= \mathrm{Bott}(z_{i,1}, \ h_{i-1,1} \circ \imath) + \mathrm{Bott}(w_{i-1}^* w_i, h_{i-1,1} \circ \imath) && \text{(e 3.90)}\\
&\quad + \mathrm{Bott}(z_i^*, \ h_{i-1,1} \circ \imath) && \text{(e 3.91)}\\
&= \kappa_{i-1} + [\Lambda_i] - \kappa_i && \text{(e 3.92)}\\
&= \alpha_{i-1} + \beta_{i-1} + [\Lambda_i] - \alpha_i - \beta_i && \text{(e 3.93)}\\
&= \alpha_{i-1} + \beta_{i-1} + [\Lambda_i] - \alpha_i - ([\Lambda_i] - \alpha_{i-1} + \alpha_i + \beta_{i-1}) && \text{(e 3.94)}\\
&= 0 && \text{(e 3.95)}
\end{aligned}$$

$i = 0, 1, 2, ..., N-1$. It follows that

$$\mathrm{Bott}(U_i, \ h_{i-1,1})|_{\mathcal{P}} = 0, \ \ i = 1, 2, ..., N-1.$$

By applying 2.11, there exists a continuous path of unitaries, $\{U_i(t) : t \in [t_{i-1}, t_i]\}$ such that

$$\begin{aligned}
U_i(t_{i-1}) = 1, \ \ U_i(t_i) = z_{i-1} w_{i-1}^* w_i z_i^* \text{ and} && \text{(e 3.96)}\\
\|U(t) h_{i-1,1}(f) U(t)^* - h_{i-1,1}(f)\| < \epsilon/32 && \text{(e 3.97)}
\end{aligned}$$

for all $f \in \mathcal{F}$ and for all $t \in [t_{i-1}, t_i]$, $i = 1, 2, ..., N$. Define $W \in B$ by

$$W(t) = w_{i-1} z_{i-1}^* U_i(t) \ \text{ for all } \ t \in [t_{i-1}, t_i], \ \ i = 1, 2, ..., N.$$

Note that $W(t_{i-1}) = w_{i-1}z_{i-1}^*$, $i = 1, 2, ..., N$, and $W(1) = w_N z_N^*$. One checks that, by (e 3.49), (e 3.96), (e 3.87), for $t \in [t_{i-1}, t_i]$,

$$\begin{aligned}
&\|W(t)\pi_t \circ \varphi(f)W(t)^* - \pi_t \circ \psi(f)\| && \text{(e 3.98)}\\
<\ &\|W(t)\pi_t \circ \varphi(f)W(t)^* - W(t)\pi_{t_{i-1}} \circ \varphi(f)W(t)^*\| && \text{(e 3.99)}\\
&+\|W(t)\pi_{t_{i-1}} \circ \varphi(f)W(t)^* - W(t)h_{i-1,1}(f)W(t)^*\| && \text{(e 3.100)}\\
&+\|W(t)h_{i-1,1}(f)W(t)^* - W(t_{i-1})h_{i-1,1}(f)W(t_{i-1})^*\| && \text{(e 3.101)}\\
&+\|W(t_{i-1})h_{i-1,1}(f)W(t_{i-1})^* - w_{i-1}h_{i-1,1}(f)w_{i-1}^*\| && \text{(e 3.102)}\\
&+\|w_{i-1}h_{i-1,1}(f)w_{i-1}^* - w_{i-1}\pi_{t_{i-1}} \circ \varphi(f)w_{i-1}^*\| && \text{(e 3.103)}\\
&+\|w_{i-1}\pi_{t_{i-1}} \circ \varphi(f)w_{i-1}^* - \pi_{t_{i-1}} \circ \psi(f)\| && \text{(e 3.104)}\\
&+\|\pi_{t_{i-1}} \circ \psi(f) - \pi_t \circ \psi(f)\| && \text{(e 3.105)}\\
<\ &\epsilon_1/4 + \epsilon_1/4 + \epsilon/32 + \delta_u + \epsilon_1/4 + \epsilon_1/4 + \epsilon_1/4 < \epsilon && \text{(e 3.106)}
\end{aligned}$$

for all $f \in \mathcal{F}$.

□

Remark 3.7. By an argument used in 5.1, we can remove the part of the assumption in (e 3.42) that $[\varphi]|_{\mathcal{P}}$ is the same as $[h]|_{\mathcal{P}}$ for some homomorphism. At present, we do not use that form of the statement.

4 Preparation for the proof

Lemma 4.1. *There is an integer $K > 0$ satisfying the following condition: Suppose that $u \in M_n(C([0,1])$ for some integer $n \geq K$. Then, for any integer $k > 0$ and any $L > 0$, if $\mathrm{cel}(u^k) \leq L$, then $\mathrm{cel}(u) \leq 2\pi/K + L/k + 6\pi$.*

Proof. (See the proof of 6.10 of [15].) It follows from Lemma 3.3 (1) of [22] that there exists a selfadjoint element $a \in M_n(C([0,1]))$ with $\|a\| \leq L$ such that

$$\det(\exp(ia)u^k)(t) = 1$$

for every $t \in [0,1]$, provided that $n \geq K$ for some integer $K \geq 1$. Fix one of such integer n. So

$$\det((\exp(ia/k)u)^k)(t) = 1$$

for all $t \in [0,1]$. This implies that, for each $t \in [0,1]$,

$$\det(\exp(ia(t)/k)u(t)) = \exp(2\pi i l(t)/k) \tag{e 4.107}$$

for some integer $l(t) \leq k$ Suppose that $b(t) = -2\pi l(t)/k$. Then $b(t)$ is a real valued continuous function on $[0,1]$, whence it is a constant. Note that

$$\exp(i(b(t)/n)\exp(ia/k) = \exp(i(b(t)/n + a/k)).$$

Then

$$\det(\exp(i(b(t)/n)\exp(ia/k)u) = 1 \tag{e 4.108}$$

(for all $t \in [0,1]$). By 3.4 and 3.1 of [22],

$$\mathrm{cel}(u) \leq 2\pi/K + L/k + 6\pi. \tag{e 4.109}$$

□

Theorem 4.2. *Let X be a compact metric space. Let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. Suppose that $\lambda : U_c(K_1(C(X))) \to \mathbb{R}_+$ is a map. There exist $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset of unitaries $\mathcal{U} \subset U_c(K_1(C(X)))$ and an integer $L > 0$ satisfying the following condition: if $\varphi, \psi : C(X) \to C([0,1], M_n)$ (for some integer $n \geq 1$) are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that*

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \ \text{ and } \ \mathrm{dist}(\overline{\langle \varphi(u) \rangle}, \overline{\langle \psi(u) \rangle}) \leq \lambda(u) \tag{e 4.110}$$

for all $u \in \mathcal{U}$, then there is a homomorphism $\Phi : C(X) \to M_L(M_n(C([0,1])))$ with finite dimensional range and a unitary $U \in M_{L+1}(M_n(C([0,1], M_n)))$ such that

$$\|U^* \mathrm{diag}(\varphi(f), \Phi(f)) U - \mathrm{diag}(\psi(f), \Phi(f))\| < \epsilon \tag{e 4.111}$$

for all $f \in \mathcal{F}$.

Proof. This follows from Theorem 3.2 of [7]. One takes $B = M_n(C([0,1]))$. Note that B has stable rank one and K_0-divisible rank T, where $T : N \times N$ is defined by $T(k,m) = [m/k] + 1$. Let K be the constant described in Lemma 3.4 of [22] (for $d = 1$). Pick a point $\xi \in X$. If $n \geq K$, we continue the argument below. If $n < K$, define $\varphi_0 : C(X) \to M_{K-n}(C([0,1])$ by $\varphi_0(f) = f(\xi)\mathrm{id}_{M_{K-n}}$ for all $f \in C(X)$. Replacing φ and ψ by $\varphi \oplus \varphi_0$ and $\psi \oplus \varphi_0$ and late absorbing φ_0, we see that we may assume that $n \geq K$.

Then, by 4.1, B has exponential divisible rank $E(L,k)$, where $E(l,k) \leq 2\pi/K + L/k + 6\pi$. It is also easy to see that $cer(B) \leq 2$. Then define $\Lambda : U(M_\infty(C(X)) \to \mathbb{R}_+$ as follows:

Let $\Pi : U(M_\infty(C(X))) \to K_1(C(X))$ be the quotient map and let $J : K_1(C(X)) \to \cup_{n=1}^\infty U(M_n(C(X)))/CU(C(X))$ be the splitting map (see 2.3). If $v \in U(M_n(C(X)))$ and $\Pi(v) \neq 0$, define $v_0 = v(J \circ \Pi(v^*))_c$, where $J \circ \Pi(v^*)_c \in U_c(K_1(C(X)))$. Define $\Lambda : U(M_\infty(C(X))) \to \mathbb{R}_+$ as follows:

$$\Lambda(v) = 2\mathrm{cel}(v) + 1 \ \text{ if } \ v \in \cup_{n=1}^\infty U_0(M_n(C(X))) \ \text{ and} \tag{e 4.112}$$

$$\Lambda(v) = \lambda(J \circ \Pi(v)_c) + 6\pi + 2\mathrm{cel}(v_0) + 1 \ \text{ if } \ \Pi(v) \neq 0, \tag{e 4.113}$$

where $\mathrm{cel}(v)$ and $\mathrm{cel}(v_0)$ is the exponential length of v and v_0 in $\cup_{n=1}^\infty U_0(M_n(C(X)))$.

Note that, for any finite subset $\mathcal{V} \subset U(M_m(C(X)))$ (for some integer $m \geq 1$), if δ is sufficiently small and $\mathcal{G}$ is sufficiently large (depends only on $\mathcal{V}$),

$$\mathrm{cel}(\varphi(v)\psi(v)^*) \leq 2\mathrm{cel}(v) + 1/4 \leq \Lambda(v) \ \text{ for all } \ v \in \mathcal{V} \ \text{ and } \ \Pi(v) = 0. \tag{e 4.114}$$

Otherwise, if $\Pi(v) \neq 0$, $v = v_c v_0$ for some $v_c \in U_c(K_1(C(X)))$ and $v_0 \in U_0(M_\infty(C(X)))$. Thus, if $v_c \in \mathcal{U}$, δ is sufficiently small and $\mathcal{G}$ is sufficiently large (depends only on $\mathcal{V}$ and $\mathcal{U}$),

$$\mathrm{cel}(\varphi(v)\psi(v^*)) = \mathrm{cel}(\varphi(v_c v_0)\psi(v_0^* v_c^*)) \tag{e 4.115}$$

$$\leq \mathrm{cel}(\varphi(v)\psi(v)^*) + 1/4 + \mathrm{cel}(\varphi(v_c)\psi(v_c^*) \tag{e 4.116}$$

$$\leq 2\mathrm{cel}(v) + 1/4 + 1/4 + \lambda(v_c) + 6\pi \leq \Lambda(v). \tag{e 4.117}$$

Therefore we can apply Theorem 3.2 of [7] directly (and the point-evaluation $f \mapsto f(\xi)\mathrm{id}_{M_k}$ will be absorbed into Φ).

□

The following is a folklore. It is a special case of Theorem 3.2 of [7]. It also follows from 2.9. We state here for the convenience for our proofs.

Lemma 4.3. *Let X be a compact metric space, let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ which forms a KL-triple for $C(X)$ and an integer N satisfying the following: Suppose that $\varphi : C(X) \to F$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map, where F is a finite dimensional C^*-algebra such that*

$$[\varphi]|_{\mathcal{P}} = [H]|_{\mathcal{P}}$$

for some unital homomorphism $H : C(X) \to M_n$. Then there exists a unital homomorphism $\Phi : C(X) \to M_N(F)$ and a unital homomorphism $h : C(X) \to M_{N+1}(F)$ such that

$$\|\varphi(f) \oplus \Phi(f) - h(f)\| < \epsilon$$

for all $f \in \mathcal{F}$.

Lemma 4.4. *Let X be a compact metric space. Let $\lambda : \cup_{n=1}^{\infty} U(M_n(C(X))) \to \mathbb{R}_+$ be a map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ and a finite subset of unitaries $\mathcal{U} \subset U_c(C(X))$, a finite subset $\{x_1, x_2, ..., x_m\} \subset X$ and an integer $L > 0$ satisfying the following condition: if φ, $\psi : C(X) \to A$ (for any unital separable simple C^*-algebra A with tracial rank at most one) are unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that*

$$[\psi]|_{\mathcal{P}} = [\varphi]|_{\mathcal{P}} \ \text{and} \tag{e 4.118}$$

$$\mathrm{dist}(\langle\varphi(u)\rangle, \langle\psi(u)\rangle) \le \lambda(u) \tag{e 4.119}$$

for all $u \in \mathcal{U}$, then, for any set of mutually orthogonal projections $p_1, p_2, ..., p_m \in M_{mL}(A)$ with $[p_i] \ge L[1_A]$, $i = 1, 2, ..., m$, there is a unitary $U \in M_{L+1}(A)$ such that

$$\|U^*\mathrm{diag}(\varphi(f), H(f))U - \mathrm{diag}(\psi(f), H(f))\| < \epsilon \tag{e 4.120}$$

for all $f \in \mathcal{F}$, where $H(f) = \sum_{i=1}^{m} f(\xi_i)p_i$ for all $f \in C(X)$.

Proof. The proof follows exactly the same way as that of 4.2. Note that it follows from [19] that $M_j(A)$ has exponential rank $1 + \epsilon$ for every integer $j \ge 1$. Also, by [15], A has stable rank one, K_0-divisible rank one, exponential divisible rank $E(L, k) = L/k + 8\pi + 1$ (see 6.10 of [15], or derive it from 4.1 directly). Thus Theorem 3.2 of [7] can also be applied as in the proof of 4.2. □

Lemma 4.5. *Let X be a compact metric space and let $s_1, s_2, ..., s_m \in \ker\rho_{C(X)}$ be a finite subset. For any $d > 0$, there is $\delta > 0$ and $\mathcal{G} \subset C(X)$ satisfying the following: For any unital C^*-algebra A with $T(A) \neq \emptyset$ and any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to A$, one has that*

$$\tau([L](s_j)) < d \ \text{ for all } \tau \in T(A), \ \ j = 1, 2, ..., m. \tag{e 4.121}$$

Proof. There is an integer $m_0 \ge 1$ and projections $p_i, q_i \in M_{m_0}(C(X))$ such that

$$[p_i] - [q_i] = s_i, \ \ i = 1, 2, ..., m.$$

Note that, for any $\tau \in T(A)$,

$$\tau(p_i) = \tau(q_i), \ \ i = 1, 2, ..., m.$$

Now suppose the lemma is false. Then there is $d_0 > 0$, a sequence of unital C^*-algebras A_n with $T(A_n) \neq \emptyset$, a sequence of δ_n-$\mathcal{G}_n$-multiplicative contractive completely positive linear

maps $L_n : C(X) \to A_n$ with $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\cup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $C(X)$ such that, for some $\tau_n \in T(A_n)$ and $1 \le j \le m$ such that

$$|\tau_n(L_n \otimes \mathrm{id}_{M_{m_0}})(p_j - q_j))| \ge d_0 \tag{e 4.122}$$

for all n. Define $L : C(X) \to \prod_{n=1}^{\infty} A_n$ by $L(f) = \{L_n(f)\}$ for all $f \in C(X)$. Let $\pi : \prod_{n=1}^{\infty} A_n \to \prod_{n=1}^{\infty} A_n / \oplus_{n=1}^{\infty} A_n$ be the quotient map. Then $\pi \circ L : C(X) \to \prod_{n=1}^{\infty} A_n / \oplus_{n=1}^{\infty} A_n$ is a unital homomorphism. Therefore, for any tracial state $t \in T(\prod_{n=1}^{\infty} A_n / \oplus_{n=1}^{\infty} A_n)$,

$$t((\pi \circ L) \otimes \mathrm{id}_{M_{m_0}})(p_j - q_j)) = 0. \tag{e 4.123}$$

Let $T_n : \prod_{n=1}^{\infty} A_n \to \mathbb{C}$ be defined by $T_n(a) = \tau_n(\pi_n(a))$ for all $a \in \prod_{n=1}^{\infty} A_n$, where $\pi_n : \prod_{n=1}^{\infty} A_n \to A_n$ is the projection to the n-coordinate. Then T_n is a tracial state. Note that, for any $a \in \oplus_{n=1}^{\infty} A_n$,

$$\lim_{n\to\infty} T_n(a) = 0. \tag{e 4.124}$$

Let T be a limit point of $\{T_n\}$. Then, by (e 4.124), T defines a tracial state on $\prod_{n=1}^{\infty} A_n / \oplus_{n=1}^{\infty} A_n$. Therefore, by (e 4.123),

$$T(((\pi \circ L_{*0}) \otimes \mathrm{id}_{M_{m_0}})(p_j - q_j) = 0.$$

It then follows that, for some subsequence $\{n_k\}$,

$$\lim_{k\to\infty} \tau_{n_k}((L_n \otimes \mathrm{id}_{M_{m_0}}(p_j - q_j)) = 0.$$

This contradicts with (e 4.122). The lemma follows.

□

When $K_i(C(X))$ $(i = 0, 1)$ is finitely generated, the following follows from 10.2 of [18]. We made a modification so it also applies to the case that $K_i(C(X))$ $(i = 0, 1)$ is not finitely generated.

Lemma 4.6. *Let X be a compact metric space. For any $\delta > 0$, any finite subset $\mathcal{G} \subset C(X)$ and any finite subset $\mathcal{P} \subset \underline{K}(C(X))$ for which the intersection of $\ker\rho_{C(X)}$ and the subgroup generated by $\mathcal{P}$ is generated by $g_1, g_2, ..., g_k$ such that $(\delta, \mathcal{G}, \mathcal{P})$ is a KL-triple, there exists an integer $N(\delta, \mathcal{G}, \mathcal{P})$ satisfies the following:*

For any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to B$, where $B = M_n$, or $B = M_n(C([0, 1])$ (for any integer $n \ge 1$) with $K = \max\{|L(g_i)| : i = 1, 2, ..., k\}$, There exists an integer $N(K) \ge 1$ satisfying the following: for any integer $N \ge N(K)/n$, there exists a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_0 : C(X) \to M_{nN} \subset M_N(B) \subset$ such that

$$\frac{N(K)}{\max\{K, 1\}} \le N(\delta, \mathcal{G}, \mathcal{P}) \text{ and} \tag{e 4.125}$$

$$([L] + [L_0])|_{\mathcal{P}} = [H]|_{\mathcal{P}} \tag{e 4.126}$$

for some unital homomorphism $H : C(X) \to M_{1+N}(B)$ with finite dimensional range.

Proof. Write $C(X) = \lim_{n\to\infty}(C(Y_n)$, where each Y_n is a finite CW complex. We use $\imath_m : C(Y_m) \to C(X)$ for the homomorphism given by the inductive limit system. Without loss of generality, we may assume that $\mathcal{G} \subset \imath_m(C(Y_m))$ for some $m \ge 1$. Let $\mathcal{G}' \subset C(Y_m)$ be a finite subset such that $\imath_m(\mathcal{G}') = \mathcal{G}$. We may further assume that $\mathcal{P} \subset [\imath_m](\underline{K}(C(Y_m)))$ and

$\mathcal{P}' \subset \underline{K}(C(Y_m))$ is a finite subset such that $[\imath_m](\mathcal{P}') = \mathcal{P}$. As defined, we also assume that $(\delta, \mathcal{G}')$ is a KK-triple for $C(Y_m)$.

Let $\imath_m(C(Y_m)) \cong C(Y)$, where Y is a compact subset of Y_m. Note that $\imath_m$ induces an embedding $\imath : C(Y) \to C(X)$. Denote by $s : X \to Y$ the surjective continuous map given by $\imath$, i.e, $\imath(f)(y) = f(s(y))$ for all $f \in C(Y)$.

Suppose that Y_m is a finite disjoint union of connected finite CW complexes $Z_1, Z_2, ..., Z_l$. One can choose $\xi_i \in Z_i$ such that $\xi_i \in Y$, $i = 1, 2, ..., l$. There are $s_1, s_2, ..., s_k \in \cup_{i=1}^l K_0(C(Z_i \setminus \{\xi_i\}))$ such that $[\imath_m](s_i) = g_i$, $i = 1, 2, ..., k$. Write $K_0(C(Z_i \setminus \{\xi_i\}) = \mathbb{Z}^{k(i)} \oplus G_i$, where G_i is the torsion subgroup. Since $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = \{0\}$, any homomorphism from $K_i(C(Y_m))$ into $K_i(\mathbb{C})$ vanishes on $Tor(K_i(C(Y_m)))$, $i = 0, 1$. To simplify the notation, without loss of generality, we may assume that $s_1, s_2, ..., s_k$ are the standard generators for $\oplus_{i=1}^l \mathbb{Z}^{(k(i)}$. We may assume that $\mathcal{G}_i' \subset C(Z_i)$ is a finite subset such that $\oplus_{i=1}^l \mathcal{G}_i' = \mathcal{P}'$ and $\mathcal{P}_i' \subset \underline{K}(C(Y_m))$ is a finite subset such that $\oplus_{i=1}^l \mathcal{P}_i' = \mathcal{P}'$.

Applying 10.2 of [18] to each component Z_i, we obtain an integer $N_i(\delta, \mathcal{G}_i', \mathcal{P}_i')$ given by 10.2 of [18]. Let $N(\delta, \mathcal{G}, \mathcal{P}) = \sum_{i=1}^l N_i(\delta, \mathcal{G}_i', \mathcal{P}_i')$.

Now let $L : C(X) \to B$ be a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map. Put $L' = L \circ \imath_m : C(Y_m) \to M_n$. Let $\kappa \in Hom_\Lambda(\underline{K}(C(Y_m)), \underline{K}(B))$ be given by L'. Let $\kappa_1 \in KK(B, \mathbb{C})$ be given by a point-evaluation, if $B = M_n(C([0,1])$, or κ_1 is given by the identity, if $B = M_n$. In either cases, one may view κ_1 is an identity on $\underline{K}(B) = \underline{K}(\mathbb{C})$. Put

$$K = \max\{|\kappa(s_j)| : j = 1, 2, ..., k\}.$$

It follows from 10.2 of [18] that there exists an integer $N(K) \geq 1$ and unital δ-$\mathcal{G}'$-multiplicative contractive completely positive linear map $L_0' : C(Y_m) \to M_{K(n)}$ such that

$$\frac{N(K)}{\max\{K, 1\}} \leq N(\delta, \mathcal{G}, \mathcal{P}) \text{ and} \tag{e 4.127}$$

$$[L_0']|_{\underline{K}(C_0(Z_i \setminus \{\xi_i\}))} = -(\kappa_1 \times \kappa)|_{\underline{K}(C_0(Z_i \setminus \{\xi_i\}))}, \ i = 1, 2, ..., l. \tag{e 4.128}$$

If $N \geq N(K)/n$, by adding some point-evaluation, if necessarily, we may assume that L_0' maps $C(Y_m)$ into a C^*-subalgebra $D \cong M_{nN}$ and D is a C^*-subalgebra of $M_N(B)$ with $1_D = 1_{M_N(B)}$. Then, viewing L_0' maps $C(Y_m)$ into $M_N(B)$,

$$\kappa + [L_0']|_{\underline{K}(C_0(Z_i \setminus \{\xi_i\}))} = 0. \tag{e 4.129}$$

There is a point-evaluation $h_0 : C(Y_m) \to M_{n+N}(B)$ at $\{\xi_1, \xi_2, ..., \xi_l\}$ such that

$$[L' \oplus L_0'] = [h_0]. \tag{e 4.130}$$

We may write

$$h_0(f) = \sum_{i=1}^l f(\xi_i) p_i \text{ for all } f \in C(Y_m),$$

where $p_1, p_2, ..., p_l$ are mutually orthogonal projections in $M_{N+1}(B)$. There is a unital contractive completely positive linear map $L_0 : C(X) \to M_N(B)$ such that

$$L_0 \circ \imath_m|_{C(Y_m)} = L_0'.$$

Note that L_0 is δ-$\mathcal{G}$-multiplicative. Define $H : C(X) \to M_{N+1}(B)$ by

$$H(f) = \sum_{i=1}^l f(s(\xi_i)) p_i \text{ for all } f \in C(X).$$

Then

$$[L \oplus L_0]|_{\mathcal{P}} = [H]|_{\mathcal{P}}.$$

□

Lemma 4.7. *Let X be a compact metric space, let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset. There exists a finite subset $\{x_1, x_2, ..., x_m\} \subset X$ ($m \geq 1$) satisfying the following: for any unital homomorphism $h_0 : C(X) \to C([0,1], M_n)$ with finite dimensional range,*

$$\|(h_0 \oplus h_1)(f) - \sum_{i=1}^{m} f(x_i)p_i\| < \epsilon \text{ for all } f \in \mathcal{F}, \tag{e 4.131}$$

where $h_1 : C(X) \to C([0,1], M_{(m-1)n})$ is a unital homomorphism with finite dimensional range and $\{p_1, p_2, ..., p_m\}$ is a set of mutually orthogonal rank n projections.

Proof. Let $\eta > 0$ such that

$$|f(x) - f(x')| < \epsilon/4 \text{ for all } f \in \mathcal{F},$$

provided that $\text{dist}(x, x') < \eta$. Let $\{x_1, x_2, ..., x_m\}$ be an η-dense subset of X. Suppose that $h_0 : C(X) \to C([0,1], M_n)$ is a unital homomorphism with finite dimensional range. Then there are $y_1, y_2, ..., y_n \in X$ and mutually orthogonal rank one projections $e_1, e_2, ..., e_n$ such that

$$h_0(f) = \sum_{i=1}^{n} f(y_i)e_i \text{ for all } f \in C(X). \tag{e 4.132}$$

Divide $\{y_1, y_2, ..., y_n\}$ into N disjoint subsets $Y_1, Y_2, ..., Y_N$ with $1 \leq N \leq m$ such that

$$\text{dist}(y_i, x_j) < \eta, \tag{e 4.133}$$

if $y_i \in Y_j$. Let $E_j = \sum_{y_i \in Y_j} e_i$ and denote by R_j the rank of E_j, $j = 1, 2, ..., N$. Choose mutually orthogonal projections $q_1, q_2, ..., q_m \in C([0,1], M_{(m-1)n})$ such that rank of $q_j = n - R_j$, $j = 1, 2, ..., N$ and rank $q_j = n$ if $N < j \leq n$. Note $\sum_{j=1}^{m} q_j = 1_{M_{(m-1)n}}$. Define $h_1 : C(X) \to C([0,1], M_{(m-1)n})$ by

$$h_1(f) = \sum_{j=1}^{n} f(x_j)q_j \text{ for all } f \in C(X).$$

Let $p_j = E_j + q_j$ if $1 \leq j \leq N$ and $p_j = q_j$ if $N < j \leq m$. Note that p_j has rank n for $j = 1, 2, ..., m$. One then checks that

$$\|(h_0 \oplus h_1)(f) - \sum_{k=1}^{m} f(x_k)p_k\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

□

Lemma 4.8. *Let X be a compact metric space, let $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset and let G be the subgroup generated by $\mathcal{P}$. Suppose $\Delta : (0,1) \to (0,1)$ is a nondecreasing function, $\eta > 0$. and $\lambda_1, \lambda_2 > 0$ are given. Suppose that $g_1, g_2, ..., g_k$ are generators of $G \cap \ker \rho_{C(X)}$.*

Suppose that $L, \Lambda : C(X) \to A$ (for some unital separable simple C^*-algebra with tracial rank at most one) are two δ-$\mathcal{G}$-multiplicative contractive completely positive linear map for which $[L](g_i)$ and $[\Lambda](g_i)$ are well defined ($i = 1, 2, ..., k$), where δ is a positive number and $\mathcal{G}$ is a finite subset of $C(X)$,

$$|\tau([L](g_i))| < \sigma \text{ and } |\tau([\Lambda](g_i))| < \sigma \text{ for all } \tau \in T(A), \ i = 1, 2, ..., k, \tag{e 4.134}$$

for some $1 > \sigma > 0$, *and*

$$\mu_{\tau\circ L}(O_r) \ge \Delta(r), \ \ \mu_{\tau\circ\Lambda}(O_r) \ge \ \Delta(r) \tag{e 4.135}$$

$$\tag{e 4.136}$$

for all $\tau \in T(A)$ *and for all* $r \ge \eta$.

Then, for any $\epsilon > 0$ *and any finite subset* $\mathcal{F}$, *any mutually orthogonal projections* $e_1, e_2, ..., e_N$, *any* $\eta > 0$, *any finite subset* $\mathcal{H} \subset A$ *and* $R_0 > 1$, *there exists a projection* $p \in A$ *and a unital* C^**-subalgebra* $B = \oplus_{j=1}^m C(X_j, M_{r(j)})$, *where* $X_j = [0,1]$, *or* X_j *is a single point, with* $1_B = p$, *mutually orthogonal projections* $e_1', e_2', ..., e_N' \in B$ *and a unital* $(\delta+\epsilon)$-$\mathcal{G}$*-multiplicative contractive completely positive linear map* $\psi_1, \psi_2 : C(X) \to B$ *such that*

$$\|L(f) - [(1-p)L(f)(1-p) + \psi_1(f)]\| < \epsilon, \tag{e 4.137}$$

$$\|\Lambda(f) - [(1-p)\Lambda(f)(1-p) + \psi_2(f)\| < \epsilon \ \textit{for all} \ f \in \mathcal{F}, \tag{e 4.138}$$

$$\tau(1-p) < \eta, \ \ \tau(e_i') \ge \min\{(1-\lambda_1)\tau(e_i) : \tau \in T(A)\} \ \textit{for all} \ \tau \in T(A) \tag{e 4.139}$$

$$r(j) \ge R_0, \ \ j = 1, 2, ..., k, \tag{e 4.140}$$

$$\|pe_ip - e_i'\| < \epsilon, t_{j,x}(e_i') \ge \min\{(1-\lambda_1)\tau(e_i) : \tau \in T(A)\} \tag{e 4.141}$$

$$|t_{j,x}([\psi_1](g_i))| < (1+\lambda_1)\sigma, |t_{j,x}([\psi_2](g_i))| < (1+\lambda_1)\sigma \tag{e 4.142}$$

$$j = 1, 2, ..., k \ \text{ and } \ x \in X_j \tag{e 4.143}$$

$$\mu_{t_{j,x}\circ\psi_1}(O_r) \ge (1-\lambda_1)\Delta(r/2(1+\lambda_2)), \ \ \mu_{t_{j,x}\circ\psi_2}(O_r) \ge (1-\lambda_1)\Delta(r/2(1+\lambda_2)) \tag{e 4.144}$$

for all $r \ge 2(1+\lambda_2)\eta$ *(We use* $t_{j,x}$ *for* $\tau_{j,x} \otimes Tr_R$ *on* $B \otimes M_R$, *where* $t_{j,x}(f) = t \circ f(x)$ *for all* $f \in C(X_j, M_{r(j)})$, *for all* $x \in X_j$ *and* t *is the normalized trace on* $M_{r(j)}$ *and* Tr_R *is the standard trace on* M_R.*)*

Moreover, for any $\epsilon_0 > 0$, *one may assume that*

$$\|pa - ap\| < \epsilon_0 \ \textit{and} \ pap \in_{\epsilon_0} B \ \textit{for all} \ a \in \mathcal{H}.$$

If furthermore, $[L]|_{\mathcal{P}} = [\Lambda]|_{\mathcal{P}}$ *(in* $KK(C(X), A)$, *then, by taking small* δ *and larger* $\mathcal{G}$, *one may further require that*

$$[\psi_1]|_{\mathcal{P}} = [\psi_2]|_{\mathcal{P}} \ \text{ in } \ KK(C(X), B).$$

Proof. The proof is a modification of that of Lemma 9.7 of [18]. We repeat many arguments here. Let $p_j, q_j \in M_R(C(X))$ such that

$$[p_j] - [q_j] = g_j, \ \ j = 1, 2, ..., k,$$

for some integer $R \ge 1$. There exists of a sequence of projections $p_n \in A$ such that

$$\lim_{n\to\infty} \|cp_n - p_nc\| = 0 \ \text{ for all } \ c \in A, \tag{e 4.145}$$

and there exists a sequence of C^*-subalgebras $B_n = \oplus_{j=1}^{m(n)} C(X_{j,n}, M_{r(j,n)})$ (where $X_{j,n} = [0,1]$ or X is a single point) with $1_{B_n} = p_n$ such that

$$\lim_{n\to\infty} \text{dist}(p_ncp_n, B_n) = 0 \ \text{ and} \tag{e 4.146}$$

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} \{\tau(1-p_n)\} = 0. \tag{e 4.147}$$

Moreover, by 3.3 of [15], we may also assume that $r(j,n) \ge R_0$ for all j. For sufficiently large n, there exists a contractive completely positive linear map $L_n' : p_nAp_n \to B_n$ such that

$$\lim_{n\to\infty} \|L_n'(a) - p_nap_n\| = 0 \ \text{ for all } \ a \in A$$

(see 2.3.9 of [11]). There are (see 2.55 and 2.5.6 of [11]) mutually orthogonal projections $e_{i,n} \in B_n$ such that

$$\lim_{n\to\infty} \|p_n e_i p_n - e'_{i,n}\| = 0, \ \ i = 1, 2, ..., N.$$

We have

$$\lim_{n\to\infty} \|L(f) - [(1-p_n)L(f)(1-p_n) + L'_n \circ L(f)]\| = 0 \text{ and} \tag{e 4.148}$$

$$\lim_{n\to\infty} \|\Lambda(f) - [(1-p_n)\Lambda(f)(1-p_n) + L'_n \circ \Lambda(f)]\| = 0 \text{ for all } f \in C(X). \tag{e 4.149}$$

Define $L'_{n,R} : M_R(A) \to M_R(A)$ by $L'_n \otimes \mathrm{id}_{M_R}$ and $L_R : M_R(C(X)) \to M_R(A)$ by $L_R = L \otimes \mathrm{id}_{M_R}$. Suppose that there exists a subsequence $\{n_k\}$, $\{j_k\}$ and $\{x_k\} \in [0, 1]$ such that

$$|t_{j_k,x_k}(L'_{n_k,R} \circ L_R(p_i - q_i))| \ge (1 + \lambda_1)\sigma \tag{e 4.150}$$

for all k. Define a state $T_k : A \to \mathbb{C}$ by $T_k(a) = t_{j_k,x_k}(a)$, $k = 1, 2, ...$ Let T be a limit point. Note $T_k(1_A) = 1$. Therefore T is a state on A. Then, by (e 4.150),

$$|T([L](g_i))| \ge (1 + \lambda_1)\sigma. \tag{e 4.151}$$

However, it is easy to check that T is a tracial state. This contradicts with (e 4.134). Put $\psi_1 = L'_n \circ L$ and $\psi_2 = L'_n \circ \Lambda$ for some large n. Then we have shown (for the choice of large n) that (e 4.137), (e 4.138) and (e 4.143) hold.

A similar argument shows that, for some sufficiently large n,

$$t_{j,x}(e'_{i,n}) \ge \min\{(1 - \lambda_1)\tau(e_i) : \tau \in T(A)\},$$

$i = 1, 2, ..., N$, for all $x \in X_j$ and $j = 1, 2, ..., m(n)$.

Moreover, a similar argument shows that, for any finitely many $f_1, f_2, ..., f_N \in C(X)$ such that $0 < f_i \le 1$, $i = 1, 2, ..., N$, we may assume (by choosing large n) that

$$t_{j,x} \circ \psi_1(f_k) \ge (1 - \lambda_1/2) \min\{\tau(L(f_k)) : \tau \in T(A)\} \text{ and} \tag{e 4.152}$$

$$t_{j,x} \circ \psi_2(f_k) \ge (1 - \lambda_1/2) \min\{\tau(\Lambda(f_k)) : \tau \in T(A)\} \tag{e 4.153}$$

for all $x \in X_j$, $j = 1, 2, ..., m$. By choosing sufficiently many (but finitely many), using the argument in the proof of 3.2, we may assume that

$$\mu_{t_{j,x}\circ\psi_i}(O_r) \ge (1 - \lambda_1)\Delta(\eta/2(1 + \lambda_2)) \tag{e 4.154}$$

for all $r \ge 2(1 + \lambda_2)\eta$ and for all $x \in X$, $j = 1, 2, ..., m$ and $i = 1, 2$.

So the first part of the lemma follows by choosing B to be B_n, p to be p_n and ψ_1 to be $L'_n \circ L$ and ψ_2 to be $L'_n \circ \Lambda$ for some sufficiently large n. Note, by (e 4.145) and (e 4.146), for any $\epsilon_0 > 0$ and any finite subset $\mathcal{H}$, we can assume that

$$\|pa - ap\| < \epsilon_0 \text{ and } pap \in_{\epsilon_0} B$$

for all $a \in \mathcal{H}$.

To see the last part of the lemma holds, taking a commutative C^*-algebra C and considering the maps $L \otimes \mathrm{id}_{M_m(C)}$ and $\Lambda \otimes \mathrm{id}_{M_m(C)}$ from $C(X)$ into $A \otimes M_m(C)$. There is $K_0 \ge 1$ such that, if $x \in Tor(K_i(C(X))) \cap G$, then $K_0 x = 0$. Let $C_1, C_2, ..., C_{K_0!}$ be unital commutative C^*-algebra such that $K_0(C_j) = \mathbb{Z} \oplus \mathbb{Z}/j\mathbb{Z}$ and $K_1(C_j) = \{0\}$, $j = 1, 2, ..., K_0!$. Let $m \ge 1$ be an integer such that a set of generators of $K_0(C(X) \otimes C_j) \cap G$ and $K_1(C(X) \otimes C_j) \cap G$ can be represented by projections and unitaries in $M_m(C(X) \otimes C_j) = C(X) \otimes C_j \otimes M_m$, $j = 1, 2, ..., K_0!$.

Choose $0 < \epsilon_1 < \epsilon$ and a finite subset $\mathcal{F}_1 \supset \mathcal{F}$ (which depends on K_0 and m above). Then one applies the first part of the lemma for this ϵ_1 and $\mathcal{F}_1$. For a finite subset of projections $E_1, E_2, ..., E_K \in C(X) \otimes C_j \otimes M_m$, if in addition that $[L]|_{\mathcal{P}} = [\Lambda]|_{\mathcal{P}}$ (with sufficiently small δ and sufficiently large $\mathcal{G}$), there are partial isometries $W_1, W_2, ..., W_K \in A \otimes C_j \otimes M_{m+R}$ for some integer $R \geq 0$ such that

$$W_i W_i^* = E_i' \oplus \mathrm{id}_{M_R(C_j)} \text{ and } W_i^* W_i = E_i'' \oplus \mathrm{id}_{M_R(C_j)},$$

where E_i' and E_i'' are two projections such that

$$\|E_i' - L \otimes \mathrm{id}_{M_m(C_j)}(E_i)\| < 1/16 \text{ and } \|E_i'' - \Lambda \otimes \mathrm{id}_{M_m(C_j)}(E_i)\| < 1/16,$$

$i = 1, 2, ..., K$.

Fix $\epsilon_0 > 0$. One then chooses a large $\mathcal{H}$ so that

$$\|pa - ap\| < \epsilon_0 \text{ and } pap \in_{\epsilon_0} B$$

imply that

$$\|(p \otimes \mathrm{id}_{M_{m+R}(C_j)})W_i - W_i(p \otimes \mathrm{id}_{M_R(C_j)})\| < \epsilon_1, \tag{e 4.155}$$
$$\|(p \otimes \mathrm{id}_{M_{m+R}(C_j)})(E_i' \otimes \mathrm{id}_{M_R(C_j)}) - (E_i' \otimes \mathrm{id}_{M_R(C_j)})(p \otimes \mathrm{id}_{M_{m+R}(C_j)})\| < \epsilon_1 \text{ and} \tag{e 4.156}$$
$$\|(p \otimes \mathrm{id}_{M_{m+R}(C_j)})(E_i'' \otimes \mathrm{id}_{M_R(C_j)}) - (E_i'' \otimes \mathrm{id}_{M_R(C_j)})(p \otimes \mathrm{id}_{M_{m+R}(C_j)})\| < \epsilon_1, \tag{e 4.157}$$
$$\|(p \otimes \mathrm{id}_{M_m(C_j)})E_i' - E_i'(p \otimes \mathrm{id}_{M_m(C_j)})\| < \epsilon_1 \text{ and} \tag{e 4.158}$$
$$\|(p \otimes \mathrm{id}_{M_m(C_j)})E_i'' - E_i''(p \otimes \mathrm{id}_{M_m(C_j)})\| < \epsilon_1 \tag{e 4.159}$$

as well as

$$(p \otimes \mathrm{id}_{M_m(C_j)})E_i'(p \otimes \mathrm{id}_{M_m(C_j)}) \in_{\epsilon_1} B \otimes M_m(C_j), \tag{e 4.160}$$
$$(p \otimes \mathrm{id}_{M_m(C_j)})E_i''(p \otimes \mathrm{id}_{M_m(C_j)}) \in_{\epsilon_1} B \otimes M_m(C_j), \tag{e 4.161}$$
$$(p \otimes \mathrm{id}_{M_{m+R}(C_j)})W_i(p \otimes \mathrm{id}_{M_{m+R}(C_i)})(p \otimes \mathrm{id}_{M_{m+R}(C_j)}) \in_{\epsilon_1} B \otimes M_{m+R}(C_j), \tag{e 4.162}$$
$$(p \otimes \mathrm{id}_{M_{m+R}(C_j)})(E_i' \otimes \mathrm{id}_{M_R(C_j)})(p \otimes \mathrm{id}_{M_{m+R}(C_j)}) \in_{\epsilon_1} B \otimes M_{m+R}(C_j) \text{ and} \tag{e 4.163}$$
$$(p \otimes \mathrm{id}_{M_{m+R}(C_j)})(E_i'' \otimes \mathrm{id}_{M_R(C_j)})(p \otimes \mathrm{id}_{M_{m+R}(C_j)}) \in_{\epsilon_1} B \otimes M_{m+R}(C_j). \tag{e 4.164}$$

It follows that (with small ϵ_1) there are projections $e_i', e_i'' \in B \otimes M_m(C_j)$ such that

$$\|e_i' - (p \otimes \mathrm{id}_{M_m(C_j)})E_i'(p \otimes \mathrm{id}_{M_m(C_j)})\| < 2\epsilon_1, \|e_i'' - (p \otimes \mathrm{id}_{M_m(C_j)})E_i''(p \otimes \mathrm{id}_{M_m(C_j)})\| < 2\epsilon_1 \text{ and}$$
$$[e_i] = [e_i'] \text{ in } K_0(B).$$

Therefore, one has

$$[\psi_1]([E_i]) = [\psi_2]([E_i]), \ \ i = 1, 2, ..., K.$$

From this, one concludes that one may require that

$$[\psi_1 \otimes \mathrm{id}_{C_j}]|_{K_0(C(X) \otimes C_j) \cap G} = [\psi_2 \otimes \mathrm{id}_{C_j}]|_{K_0(C(X) \otimes C_j) \cap G},$$

$j = 1, 2, ..., K_0!$. A similar argument shows that one may also require that

$$[\psi_1 \otimes \mathrm{id}_{C_j}]|_{K_1(C(X) \otimes C_j) \cap G} = [\psi_2 \otimes \mathrm{id}_{C_j}]|_{K_1(C(X) \otimes C_j) \cap G},$$

$j = 1, 2, ..., K_0!$. It follows that one may require that

$$[\psi_1] = [\psi_2] \text{ in } KK(C(X), B).$$

□

5 The main results

Lemma 5.1. *Let X be a compact metric space, let $\epsilon > 0$, $\epsilon_0 > 0$, let $\{x_1, x_2, ..., x_m\} \subset X$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta : (0,1) \to (0,1)$ be an increasing map with $\lim_{t\to 0} \Delta(t) = 0$. Let $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset, $K \geq 1$ be an integer and let $\eta_0 > 0$. Then, there exists $\eta > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: For any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $L, \Lambda : C(X) \to A$ for some unital separable simple C^*-algebra A with tracial rank at most one for which*

$$[\Lambda]|_{\mathcal{P}} = [L]|_{\mathcal{P}} \text{ and } \mu_{\tau\circ L}(O_r), \mu_{\tau\circ L}(O_r) \geq \Delta(r) \tag{e 5.165}$$

for all open balls O_r with radius $1 > r \geq \eta$, and, for any $\epsilon_{00} > 0$ and any finite subset $\mathcal{H} \subset A$, there exist mutually orthogonal projections $P_1, P_2, P_3, p_1, p_2, ..., p_m \in A$ with $P_1 \oplus P_2 \oplus P_3 \oplus \sum_{i=1}^m p_i = 1_A$,

$$\tau(P_3) > 1 - \epsilon_0 \text{ for all } \tau \in T(A) \text{ and} \tag{e 5.166}$$

$$K[P_1 \oplus P_2] \leq [p_i], \quad i = 1, 2, ..., m, \tag{e 5.167}$$

and there exists a unital ϵ-$\mathcal{F}$-multiplicative contractive completely positive linear map $\psi : C(X) \to P_2BP_2$ whose range contained in a finite dimensional C^*-subalgebra, unital ϵ-$\mathcal{F}$-multiplicative contractive completely positive linear maps $H_1, H_2 : C(X) \to P_3BP_3 \subset P_3AP_3$, where $P_2, P_3 \in B$, $B = \oplus_{j=1}^N B_j$, and $B_j = C(X_j, M_{r(j)})$ ($X_j = [0,1]$, or X_j is a point) with

$$[H_1]|_{\mathcal{P}} = [H_2]|_{\mathcal{P}} = [h_0]|_{\mathcal{P}},$$

for some unital homomorphism $h_0 : C(X) \to C$, where $C = P_3BP_3$, $C = \oplus_{j=1}^N C_j$ and $C_j = C(X_j, M_{r'(j)})$, and a unitary $W \in A$ such that

$$\|L(f) - [(P_1L(f)P_1 \oplus H_1(f) \oplus \psi(f) \oplus \sum_{i=1}^m f(x_i)p_i]\| < \epsilon \tag{e 5.168}$$

$$\|\mathrm{Ad}\, W \circ \Lambda(f) - [P_1(\mathrm{Ad}\, W \circ \Lambda)(f)P_1 \oplus H_2(f) \oplus \psi(f) \oplus \sum_{i=1}^m f(x_i)p_i]\| < \epsilon \tag{e 5.169}$$

$$\text{for all } f \in \mathcal{F}, \tag{e 5.170}$$

$$\mu_{t_{j,x}\circ H_i}(O_r) \geq \Delta(r/3)/2 \text{ and } t(P_2 + \sum_{i=1}^m p_i) < \epsilon_0 \tag{e 5.171}$$

for all $r \geq \eta_0$, $x \in X_j$, where $t_{j,x}$ is the composition of the point-evaluation at x and the normalized trace on $M_{r'(j)}$, $j = 1, 2, ..., k$, and for all $t \in T(B)$, and

$$\|P_1a - aP_1\| < \epsilon_{00}, \ (1 - P_1)a(1 - P_1) \in_{\epsilon_{00}} B \text{ for all } a \in \mathcal{H} \cup L(\mathcal{F}) \cup \Lambda(\mathcal{F}), \tag{e 5.172}$$

where $1_B = 1 - P_1$, Moreover,

$$[P_1LP_1]|_{\mathcal{P}} = [P_1(\mathrm{Ad}\, W \circ \Lambda)P_1]|_{\mathcal{P}}. \tag{e 5.173}$$

Proof. Let ϵ, ϵ_0, $\{x_1, x_2, ..., x_m\} \subset X$, a finite subset $\mathcal{F} \subset X$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, Δ, $K \geq 1$ and $\eta_0 > 0$ be as described. We may assume that $(\epsilon, \mathcal{F}, \mathcal{P})$ is a KL-triple for $C(X)$ and $0 < \epsilon_0, \epsilon < 1/16$.

Let $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset C(X)$ be a finite subset (in place of $\mathcal{G}$), $\mathcal{P}_1 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) be a finite subset and K_1 be an integer (in place of L) for $\min\{\epsilon/16, \epsilon_0/16\}$ and $\mathcal{F}$ required by 4.3.

We may also assume, without loss of generality, that $\mathcal{P} \subset \mathcal{P}_1$ and $(\delta_1, \mathcal{G}_1, \mathcal{P}_1)$ forms a KL-triple. We may further assume that, if $L', L'' : C(X) \to C$ (for any unital C^*-algebra C) are ϵ-$\mathcal{G}_1$-multiplicative contractive completely positive linear maps and

$$\|L'(f) - L''(f)\| < \epsilon \text{ for all } f \in \mathcal{G}_1,$$

then

$$[L']|_{\mathcal{P}_1} = [L'']|_{\mathcal{P}_1}.$$

Let G be the subgroup generated by $\mathcal{P}_1$ and let $s_1, s_2, ..., s_{k_0}$ be a set of generators of $G \cap \ker\rho_{C(X)}$.

Let $\epsilon_2 = \min\{\epsilon/64, \epsilon_0/64, \delta_1/2, \delta'/2\}$ and $\mathcal{G}_2 = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}'$.

Let $N_1 = N(\epsilon_2, \mathcal{G}_2, \mathcal{P}_1)$ be as in 4.6 where δ is replace by ϵ_2, $\mathcal{G}$ is replace by $\mathcal{G}_2$ and $\mathcal{P}$ is replaced by $\mathcal{P}_1$.

Let N_2 (in place m) and $\{y_1, y_2, ..., y_{N_2}\}$ (in place of $\{x_1, x_2, ..., x_m\}$) be as in 4.7 for ϵ_2 (in place ϵ) and $\mathcal{G}_2$ (in place of $\mathcal{F}$). One may assume that $N_2 > m$ and $y_j = x_j$, $j = 1, 2, ..., m$.

Choose $\eta' > 0$ satisfying the following:

$$|f(x) - f(x')| < \epsilon_2/16$$

for all $f \in \mathcal{G}_2$, if $\text{dist}(x, x') < 2\eta'$. Moreover, we may assume that

$$O_{4\eta'}(y_j) \cap O_{4\eta'}(y_i) = \emptyset \text{ if } i \neq j.$$

Choose $\eta'' > 0$ such that $\eta'' < \eta_0/4$ and

$$\Delta(\eta'') < \frac{\Delta(\eta_0/4)}{256N_2}.$$

Choose

$$\eta = \min\{\eta_1/4, \eta_2/4, \eta'/4, \eta''/4\}.$$

Let $\delta_2 > 0$ (in place of δ) and let $\mathcal{G}_3 \subset C(X)$ be a finite subset required by Lemma 9.6 of [18] for $\epsilon_2/2$ (in place of ϵ), $\mathcal{G}_2$ (in place of $\mathcal{F}$), η and $1/256$ (in place of r).

Choose $\bar{K}$ to be an integer which is greater than the integer K given by the lemma. We may assume that $K > 4$. Choose $d > 0$ such that

$$16d\bar{K}N_1N_2^2(K_1 + 1) < \epsilon_0\Delta(\eta)/2^{11}. \tag{e 5.174}$$

It follows from 4.5 that there are $\delta_3 > 0$ and a finite subset $\mathcal{G}_4 \subset C(X)$ such that, for any unital δ_3-$\mathcal{G}_4$-multiplicative contractive completely positive linear map $\Psi : C(X) \to C$ (for any unital C^*-algebra C with $T(C) \neq \emptyset$),

$$|\tau([\Psi](s_i))| < d/8 \text{ for all } \tau \in T(C).$$

Let $\delta = \min\{\epsilon_2/4, \delta_2/4, \delta_3/4, \delta_4\}$ and $\mathcal{G} = \cup_{i=1}^5 \mathcal{G}_i$.

Now suppose that $L, \Lambda : C(X) \to A$, where A is a unital simple C^*-algebra with tracial rank at most one, satisfy the assumptions of the theorem for the above δ, $\mathcal{G}$, η and Δ.

By Theorem 9.6 of [18] and by the choice of η, there exists projections $Q_1, Q_2 \in A$ and two sets of mutually orthogonal projections $\{E_1, E_2, ..., E_{N_2}\}$ in $(1 - Q_1)A(1 - Q_1)$ and $\{E_1', E_2', ..., E_{N_2}'\}$ in $(1 - Q_2)A(1 - Q_2)$ such that $\sum_{i=1}^{N_2} E_i = 1 - Q_1$, $\sum_{i=1}^{N_2} E_i' = (1 - Q_2)$,

$$\|L(g) - [Q_1L(g)Q_1 \oplus \sum_{i=1}^{N_2} g(y_i)E_i)\| < \epsilon_2/2, \tag{e 5.175}$$

$$\|\Lambda(g) - [Q_2\Lambda(g)Q_2 \oplus \sum_{i=1}^{N_2} g(y_i)E_i'\| < \epsilon_2/2 \tag{e 5.176}$$

for all $g \in \mathcal{G}_2$,

$$\Delta(\eta) > \tau(E_i) \ge (1 - 1/2^{12})\Delta(\eta) \text{ and} \tag{e 5.177}$$
$$\Delta(\eta) > \tau(E_i') \ge (1 - 1/2^{12})\Delta(\eta), \ \ i = 1, 2, ..., N_2 \tag{e 5.178}$$

for all $\tau \in T(A)$. Since A has tracial rank at most one (see Lemma 9.9 of [18]), there is, for each i, a projection $q_i \le E_i$ such that $[q_i] \le [E_i']$ and

$$\tau(q_i) \ge (1 - 1/2^{11})\Delta(\eta) \text{ for all } \tau \in T(A).$$

Let $q_i' \le E_i'$ such that $[q_i] = [q_i']$, $i = 1, 2, ..., N_2$. Let $Q_0 = 1 - \sum_{i=1}^{N_2} q_i$ and $Q_0' = 1 - \sum_{i=1}^{N_2} q_i'$. Then, we have that

$$\|L(g) - [Q_1 L(g) Q_1 \oplus \sum_{i=1}^{N_2} g(y_i)(E_i - q_i) + \sum_{i=1}^{N_2} g(y_i) q_i]\| < \epsilon_2/2 \tag{e 5.179}$$
$$\|\Lambda(g) - [Q_2 \Lambda(g) Q_2 \oplus \sum_{i=1}^{N_2} g(y_i)(E_i' - q_i') + \sum_{i=1}^{N_2} g(y_i) q_i'\| < \epsilon_2/2 \tag{e 5.180}$$

for all $g \in \mathcal{G}_2$. Since $[q_i] = [q_i']$, $i = 1, 2, ..., N_2$, there is a unitary $W_1 \in A$ such that

$$W_1^* q_i' W_1 = q_i, \ \ i = 1, 2, ..., N_2 \text{ and } W_1^* Q_0' W_1 = Q_0.$$

Define $L_1 : C(X) \to Q_0 A Q_0$ by $L_1(f) = Q_1 L(f) Q_1 \oplus \sum_{i=1}^{N_2} f(y_i)(E_i - q_i)$ for all $f \in C(X)$. Define $\Lambda_1 : C(X) \to Q_0 A Q_0$ by $\Lambda_1(f) = W_1^*(Q_2 \Lambda(f) Q_1 \oplus \sum_{i=1}^{N_2} f(y_i)(E_i' - q_i'))W_1$ for all $f \in C(X)$. Then L_1 and Λ_1 are ϵ_2-$\mathcal{G}_4$-multiplicative and

$$\|L(g) - [L_1(g) + \sum_{i=1}^{N_2} g(y_i) q_i]\| < \epsilon_2/2 \text{ and} \tag{e 5.181}$$
$$\|\mathrm{Ad}\, W_1 \circ \Lambda(g) - [\Lambda_1(g) + \sum_{i=1}^{N_2} g(y_i) q_i\| < \epsilon_2/2 \tag{e 5.182}$$

for all $g \in \mathcal{G}_2$. Note that

$$[L_1](s_i) = [L](s_i) \text{ and } [\Lambda_1](s_i) = [\Lambda](s_i), \ \ i = 1, 2, ..., k_0. \tag{e 5.183}$$

We compute that

$$\mu_{\tau \circ Q_1 L Q_1}(O_r) \ge \Delta(r) - N_2 \Delta(\eta) \ge 255\Delta(r)/256$$

for all $\tau \in T(A)$ and $r \ge \eta_0/4$. It follows that

$$\mu_{\tau \circ L_1}(O_r) \ge 255\Delta(r)/256 \tag{e 5.184}$$

for all $\tau \in T(A)$ and $r \ge \eta_0/4$. Similarly,

$$\mu_{\tau \circ \Lambda_1}(O_r) \ge 255\Delta(r)/256 \tag{e 5.185}$$

for all $\tau \in T(A)$ and $r \ge \eta_0/4$.

Let $\theta < \frac{\epsilon_0 \Delta(\eta/2)}{8 N_1 N_2 K}$. Let $\epsilon_{00} > 0$ and let $\mathcal{H} \subset A$ be a finite subset. Define

$$\mathcal{H}_1 = \mathcal{H} \cup L(\mathcal{G}) \cup L_1(\mathcal{G}) \cup \Lambda_1(\mathcal{G}) \cup \mathrm{Ad}\, W_1 \circ \Lambda(\mathcal{G}) \cup \{P_1, q_1, q_2, ..., q_{N_2}\}.$$

Let $0 < \delta_0 < \min\{\epsilon_2/2, \delta/4, \epsilon_{00}\}$ and put

$$L'(f) = L_1(f) \oplus \sum_{i=1}^{N_2} f(y_i)q_i \text{ and} \tag{e 5.186}$$

$$\Lambda'(f) = \Lambda_1(f) \oplus \sum_{i=1}^{N_2} f(y_i)q_i \tag{e 5.187}$$

for all $f \in C(X)$. Since A has tracial rank at most one, by 4.8, there exists a projection $Q_3 \in A$ and $B = \oplus_{j=1}^{N} B_j$ with $1_B = Q_3$, where $B_j = M_{r(j)}(C(X_j))$ and $X_j = [0,1]$, or X_j is a point, such that

$$\|Q_3 a - aQ_3\| < \delta_0, \ \ Q_3 a Q_3 \in_{\delta_0} B \text{ for all } a \in \mathcal{H}_1, \tag{e 5.188}$$

$$\|L'(f) - [(1-Q_3)L'(f)(1-Q_3) \oplus L_3(f)]\| < \epsilon_2/2N_2, \tag{e 5.189}$$

$$\|\Lambda'(f) - [(1-Q_3)\Lambda(f)(1-Q_3) \oplus \Lambda_3(f)]\| < \epsilon_2/2N_2 \text{ for all } f \in \mathcal{G}, \tag{e 5.190}$$

$$\tau(1-Q_3) < \theta \text{ for all } \tau \in T(A), \tag{e 5.191}$$

$$|T_{j,x}([L_3](s_i))| < (1+1/128)(d/8), \ \ |T_{j,x}([\Lambda_3](s_i)| < (1+1/128)(d/8) \tag{e 5.192}$$

$$\mu_{T_{j,x}\circ L_3}(O_r) \ge 3\Delta(r/3)/4, \ \ \mu_{T_{j,x}\circ \Lambda_3}(O_r) \ge 3\Delta(r/3)/4 \tag{e 5.193}$$

for all $r \ge \eta_0$, and for all j and $x \in X_j$, where $T_{j,x}$ is the normalized trace of $M_{r(j)}$ at $x \in X_j$, and

$$r(j) > \frac{2^{15}\bar{K}N_1N_2^2(K_1+1)}{\epsilon_0\Delta(\eta/2)}, \ \ j = 1,2,...,N. \tag{e 5.194}$$

Moreover,

$$[L_3]|_{\mathcal{P}_1} = [\Lambda_3]|_{\mathcal{P}_1} \text{ in } KL(C(X), B). \tag{e 5.195}$$

Therefore

$$\|Q_3L'(f)Q_3 - L_3(f)\| < \epsilon_2/2N_2 \text{ and } \|Q_3\Lambda'(f)Q_3 - \Lambda_3(f)\| < \epsilon_2/2N_2 \tag{e 5.196}$$

for all $f \in \mathcal{G}$. By 4.8, we further obtain mutually orthogonal projections $e_1, e_2, ..., e_{N_2} \in B$ such that

$$\|L_3(f) - [EL_3(f)E \oplus \sum_{i=1}^{N_2} f(y_i)e_i]\| < \epsilon_2/2 \tag{e 5.197}$$

$$\|\Lambda_3(f) - [E\Lambda_3(f)E \oplus \sum_{i=1}^{N_2} f(y_i)e_i]\| < \epsilon_2/2 \tag{e 5.198}$$

for all $f \in \mathcal{G}$, where $E = 1_B - \sum_{i=1}^{N_2} e_i$. Moreover we require that

$$\Delta(\eta)/2 > T_{j,x}(e_i) \ge (1-1/250)\Delta(\eta), \ \ \Delta(\eta)/2 > \tau(e_i) \ge (1-1/250)\Delta(\eta) \tag{e 5.199}$$

for all $x \in X_j$, where $T_{j,x}$ is the normalized trace evaluated at x, and for all $\tau \in T(A)$, $j = 1,2,...,N_2$.

Define $L_4 = EL_3E$ and $\Lambda_4 = E\Lambda_3E$. We compute, by (e 5.193) and (e 5.199)

$$\mu_{T_{j,x}\circ L_4}(O_r) \ \ge \ 3\Delta(r/3)/4 - N_2\Delta(\eta) \ge \Delta(r/3)/2 \text{ and} \tag{e 5.200}$$

$$\mu_{T_{j,x}\circ \Lambda_4}(O_r) \ \ge \ \Delta(r/3)/2. \tag{e 5.201}$$

for all $r \ge \eta_0$ and for all $x \in X_j$ and $j = 1, 2, ..., N$.

We compute that

$$[L_4](s_j) = [\Lambda_4](s_j) = [L](s_j) = [\Lambda](s_j), \ \ j = 1, 2, ..., k_0. \tag{e 5.202}$$

Put $C_j' = E(M_{r(j)}(C(X_j))E = M_{r''(j)}(C(X_j))$, where $r''(j) \le r(j)$, and Put $L_{4,j} = \pi_j \circ L_4$ and $\Lambda_{4,j} = \pi_j \circ \Lambda_4$, where $\pi_j : EBE \to EB_jE$ is the projection, $j = 1, 2, ..., N$.

Put

$$d_j = \max\{|T_{j,x}([L_4](s_k))| : k = 1, 2, ..., k_0\},$$

$j = 1, 2, ..., N$. Note that $d_j \le d/4$, $j = 1, 2, ..., N$.

It follows from 4.6 that there exist unital ϵ_2-$\mathcal{G}_2$-multiplicative $L_{0,j}, \bar{L}_{0,j} : C(X) \to M_{J_j}(C(X_j))$ whose ranges are contained in finite dimensional C^*-subalgebras, where

$$J_j = d_j N_1 r(j) \le dN_1 r(j) \tag{e 5.203}$$

such that

$$[L_{4,j} \oplus L_{0,j}]|_{\mathcal{P}_1} = [H_{0,j}]|_{\mathcal{P}_1} \ \text{ and } \ [L_{0,j} \oplus \bar{L}_{0,j}]|_{\mathcal{P}_1} = [h_{0,j}]|_{\mathcal{P}_1} \tag{e 5.204}$$

for some unital homomorphisms $H_{0,j} : C(X) \to M_{r''(j)+J_j}(C(X_j))$ and $h_{0,j} : C(X) \to M_{2J_j}(C(X_j))$ with finite dimensional range.

By applying 4.3, we obtain a unital homomorphism $h_{1,j} : C(X) \to M_{2J_jK_1}(C(X_j))$ with finite dimensional range and a unital homomorphism $H_{1,j} : C(X) \to M_{2J_j(K_1+1)}(C(X_j))$ with finite dimensional range such that

$$\|(L_{0,j} \oplus \bar{L}_{0,j} \oplus h_{1,j})(f) - H_{1,j}(f)\| < \min\{\epsilon/16, \epsilon_0/16\} \ \text{ for all } \ g \in \mathcal{F}. \tag{e 5.205}$$

It follows from 4.7 that there are mutually orthogonal rank $2J_j(K_1 + 1)$ projections $q_{i,j}' \in M_{N_2 2J_1(K_1+1)}(C(X_j))$ and a unital homomorphism $h_{2,j} : C(X) \to M_{(N_2-1)(2J_1(K_1+1)}(C_j)$ with finite dimensional range such that

$$\|H_{1,j}(f) \oplus h_{2,j}(f) - \sum_{i=1}^{N_2} f(y_i) q_{i,j}'\| < \epsilon_2 \ \text{ for all } \ f \in \mathcal{F}, \tag{e 5.206}$$

$j = 1, 2, ..., N$.

There is, for each i and j, by (e 5.199) and (e 5.174), a projection $p_{i,j}' \le e_i$ such that

$$\epsilon_0/128N_2 \ge T_{j,x}(p_{i,j}') \ge \epsilon_0/256N_2 \ge 16d\bar{K}N_1N_2(K_1 + 1) \tag{e 5.207}$$

for $x \in X_j$, $j = 1, 2, ..., N$ and $i = 1, 2, ..., N_2$.

Put $L_0 = \sum_{j=1}^N L_{0,j}$, $\bar{L}_0 = \sum_{j=1}^N \bar{L}_{0,j}$, $h_i = \sum_{j=1}^N h_{i,j}$, $i = 1, 2$. There is a projection $q_{i,j} \le p_{i,j}'$ in B_j such that $[q_{i,j}] = [q_{i,j}']$ in $K_0(B_j)$, $j = 1, 2, ..., N$. Put $p_i'' = \sum_{j=1}^N q_{i,j}$, $i = 1, 2, ...N_2$. Then

$$T_{j,x}(p_i'') < 2J_j(K_1 + 1)/r(j) \ \text{ for all } \ x \in X_j. \tag{e 5.208}$$

Thus we obtain a unitary $W_0 \in B$ such that

$$\|\mathrm{Ad}\, W_0 \circ (L_0 \oplus \bar{L}_0 \oplus h_1 \oplus h_2)(f) - \sum_{i=1}^{N_2} f(y_i) p_i''\| < \epsilon_2 + \epsilon/16 \tag{e 5.209}$$

for all $f \in \mathcal{F}$.

Now define

$$H_1(f) = L_4(f) \oplus \mathrm{Ad}\, W_0 \circ L_0 \oplus h_1(f) \oplus h_2(f) \oplus \sum_{k=1}^{m} f(y_k)(e_i - p_i') \oplus \sum_{j=m+1}^{N_2} f(y_j)(e_i - p_i'')$$

for all $f \in C(X)$, $\psi = \mathrm{Ad}\, W_0 \circ \bar{L}_0$. Let $P_1 = (1 - Q_3)$, $P_2 = W_0^* \bar{L}_0(1_{C(X)})W_0$, $P_3 = H_1(1_{C(X)})$. $p_j = p_j' - p_j''$, $j = 1, 2, ..., N_2$. Then we estimate that, by (e 5.197) and (e 5.209),

$$\|L_3(f) - (H_1 \oplus \psi(f) \oplus \sum_{j=1}^{m} f(x_j)p_j)\| \tag{e 5.210}$$

$$< \quad \|L_3(f) - (L_4(f) \oplus \sum_{i=1}^{N_2} f(y_j)e_j)\| + \tag{e 5.211}$$

$$\|L_4(f) \oplus \sum_{j=1}^{N_2} f(y_j)e_j - L_4(f) \oplus \mathrm{Ad}\, W_0(L_0 \oplus \bar{L}_0 \oplus h_1 \oplus h_2)(f) \oplus \sum_{j=1}^{m} f(y_j)p_j)\| \tag{e 5.212}$$

$$< \quad \epsilon_2/2 + \epsilon_2 + \epsilon/16 = 3\epsilon_2/2 + \epsilon/16 \tag{e 5.213}$$

for all $f \in \mathcal{F}$. It follows from (e 5.181), (e 5.186), (e 5.189) and (e 5.213) that

$$\|L(f) - [P_1 L(f) P_1 \oplus \psi(f) \oplus \sum_{j=1}^{m} f(x_j)p_j \oplus H_1(f)\| \tag{e 5.214}$$

$$< \quad \|L(f) - L'(f)\| + \|L'(f) - (1 - Q_3)L'(f)(1 - Q_3) \oplus L_3(f)\| \tag{e 5.215}$$

$$+\|(1 - Q_3)L'(f)(1 - Q_3) \oplus L_3(f) - P_1 L(f) P_1 \oplus L_3(f)\| \tag{e 5.216}$$

$$+\|P_1 L(f) P_1 \oplus L_3(f) - [P_1 L(f) P_1 \oplus H_1(f) \oplus \psi(f) \oplus \sum_{j=1}^{m} f(x_j)p_j]\| \tag{e 5.217}$$

$$< \quad \epsilon_2/2 + \epsilon_2/2 + 3\epsilon_2/2 + \epsilon/16 < \epsilon \tag{e 5.218}$$

for all $f \in \mathcal{F}$. Define

$$H_2(f) = \Lambda_4 \oplus \mathrm{Ad}\, W_0 \circ L_0 \oplus h_1(f) \oplus h_2(f) \oplus \sum_{k=1}^{N_2} f(y_k)(e_i - p_i') \oplus \sum_{j=m+1}^{N_2} f(y_j)(e_j - p_j'').$$

Similarly, we also have

$$\|\mathrm{Ad}\, W_1 \circ \Lambda(f) - [P_1(\mathrm{Ad}\, W_1 \circ \Lambda(f))P_1 \oplus \psi(f) \oplus \sum_{j=1}^{m} f(x_j)p_j \oplus H_2(f)]\| < \epsilon \tag{e 5.219}$$

for all $f \in \mathcal{F}$.

Note also that, (by (e 5.174) and (e 5.208))

$$\frac{\epsilon_0 \Delta(\eta)}{128 N_2} \geq T_{j,x}(p_i) \quad = \quad T_{j,x}(p_j' - p_j'') \geq \frac{\epsilon_0 \Delta(\eta)}{256 N_2} - 2J_j(K_1 + 1)/r(j) \tag{e 5.220}$$

$$\geq \quad 14\bar{K} d N_1 N_2 (K_1 + 1) \tag{e 5.221}$$

for all $\tau \in T(A)$ and $i = 1, 2, ..., N_2$. Therefore, by (e 5.191),

$$\tau(p_i) \geq (1 - \theta)14 d\bar{K} N_1 N_2 (K_1 + 1) \text{ for all } \tau \in T(A). \tag{e 5.222}$$

Note that, by (e 5.203),

$$T_{j,x}(P_2) \le d_j N_1 \text{ and } \tau(P_1) = \tau(1-Q_3) < \theta \tag{e 5.223}$$

for all $x \in X_j$, $j = 1, 2, ..., N$ and for all $\tau \in T(A)$. It follows that

$$\tau(p_i) > K\tau(P_2) + K\tau(P_1) \text{ for all } \tau \in T(A), \ i = 1, 2, ..., m. \tag{e 5.224}$$

This gives (e 5.167). To obtain (e 5.166), we note, by (e 5.207) and (e 5.174) that

$$\tau(P_1) + \tau(P_2) + \sum_{i=1}^{m} \tau(p_i) < \theta + dN_1 + \frac{\epsilon_0}{128N_2} m \tag{e 5.225}$$

$$< \frac{\epsilon_0 \Delta(\eta/2)}{4N_1N_2\tilde{K}} + \frac{\epsilon_0\Delta(\eta/2)}{256\tilde{K}} + \frac{\epsilon_0}{128} < \epsilon_0/2 \tag{e 5.226}$$

for all $\tau \in T(A)$. We also have that

$$T_{j,x}(P_2) + \sum_{i=1}^{m} T_{j,x}(p_i) < \epsilon_0/2 \text{ for all } x \in X_j. \tag{e 5.227}$$

Thus

$$t(P_2 + \sum_{i=1}^{m} p_i) < \epsilon_0/2$$

for all $t \in T(B)$. This implies (e 5.166). We write $C = P_3BP_3 = \oplus_{j=1}^{N} C_j$, where $C_j = M_{r'(j)}(C(X_j))$, $j = 1, 2, ..., N$. Finally, from (e 5.200) and (e 5.201),

$$\mu_{t_{j,x}\circ H_i}(O_r) \ge \Delta(r/3)/2 \tag{e 5.228}$$

for all $r \ge \eta_0$ and $x \in X_j$, where $t_{j,x}$ is the normalized trace of $M_{r'(j)}$ evaluated at $x \in X_j$, $j = 1, 2, ..., N$ and $i = 1, 2$. The lemma follows.

□

Lemma 5.2. *Let C be a separable unital C^*-algebra with $T(C) \neq \emptyset$, let $\mathcal{U} \subset U_c(K_1(C))$ be a finite subset, $\mathcal{F} \subset C$ be a finite subset and let $\lambda > 0$. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following: Suppose that $L_1, L_2 : C \to A$ (for some unital C^*-algebra A) are two δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that*

$$\mathrm{dist}(\overline{\langle L_1(u)\rangle}, \overline{\langle L_2(u)\rangle}) \le \Gamma \tag{e 5.229}$$

for all $u \in \mathcal{U}$ and for some $\Gamma > 0$. There exists a finite subset $\mathcal{H} \subset A$ and $\sigma > 0$ such that, if $p \in A$ is a projection such that

$$\|pa - ap\| < \sigma, pap \in_\sigma B \text{ for all } a \in \mathcal{H},$$

where $1_B = p$ and $B \subset pAp$ is a unital C^*-subalgebra, and $\Lambda_1, \Lambda_2 : C \to B$ are two 2δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that

$$\|pL_i(g)p - \Lambda_i(g)\| < \sigma \text{ for all } g \in \mathcal{G},$$

then

$$\mathrm{dist}(\overline{\langle \Lambda_1(u)\rangle}, \overline{\langle \Lambda_2(u)\rangle}) \le \Gamma + \lambda$$

for all $u \in \mathcal{U}$.

Moreover,

$$|\tau \circ \Lambda_1(f) - \tau \circ \Lambda_2(f)| \le \lambda + \max\{|t \circ L_1(f) - \tau \circ L_2(f)| : f \in \mathcal{F}, \ t \in T(A)\} \tag{e 5.230}$$

for all $f \in \mathcal{F}$ and $\tau \in T(B)$.

Proof. Let $\mathcal{U}$ and $\mathcal{F}$ be fixed. Then there is an integer $k \ge 1$ such that every element in $\mathcal{U}$ is represented by a unitary in $M_k(C)$. To simplify notation, replacing A by $M_k(A)$, replacing B by $M_k(B)$, and later replacing L_i by $L_i \otimes \mathrm{id}_{M_k}$, $i = 1, 2$, without loss of generality, we may assume that $\mathcal{U}$ is actually in $U(A)$. We choose δ and $\mathcal{G}$ such that for any 2δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps L from C, $\langle L(u)\rangle$ is well defined for all $u \in \mathcal{U}$.

Now let L_1 and L_2 be as described (for the above choice of δ and $\mathcal{G}$). Suppose that $\mathcal{U} = \{u_1, u_2, ..., u_m\}$. Then there are $w_1, w_2, ..., w_m \in CU(A)$ such that

$$\|\langle L_1(u_i)\rangle\langle L_2(u_i^*)\rangle - w_i\| \le \Gamma + \lambda/2.$$

It is clear that, if $\mathcal{H}$ is sufficiently large (containing at least $L_1(u)$ and $L_2(u)$ for all $u \in \mathcal{U}$ and many other elements in $U(A)$) and σ is sufficiently small, $\langle pL_ip(u_j)\rangle$ are well defined and

$$\|\langle pL_ip(u_j)\rangle - \langle \Lambda_i(u_j)\rangle\| < \lambda/16$$

($j = 1, 2, ..., m$ and $i = 1, 2$) and there are unitaries $v_1, v_2, ..., v_m \in CU(B)$ such that

$$\|pw_ip - v_i\| < \lambda/16, \ \ i = 1, 2, ..., m.$$

It follows that

$$\mathrm{dist}(\overline{\langle \Lambda_1(u)\rangle}, \overline{\langle \Lambda_2(u)\rangle}) \le \Gamma + \lambda \tag{e 5.231}$$

for all $u \in \mathcal{U}$.

Similarly, for each $f \in \mathcal{F}$, there are $x_1(f), x_2(f), ..., x_{f(m)}(f) \in A$ such that

$$\|L_1(f) - \sum_{i=1}^{f(m)} x_i(f)^*x_i(f)\| < \lambda/8 \ \text{ and} \tag{e 5.232}$$

$$\|L_2(f) - \sum_{i=1}^{f(m)} x_i(f)x_i(f)^*\| < M + \lambda/8 \tag{e 5.233}$$

where $M = \max\{|\tau \circ L_1(f) - \tau \circ L_2(f)| : f \in \mathcal{F}, \ \tau \in T(A)\}$ (see [1]). We compute that, with sufficiently large $\mathcal{H}$ and small σ, there are $y_1(f), y_2(f), ..., y_{f(m)}(f) \in B$ such that

$$\|\Lambda_1(f) - \sum_{i=1}^{f(m)} y_i(f)^*y_i(f)\| < \lambda/4 \ \text{ and} \tag{e 5.234}$$

$$\|\Lambda_1(f) - \sum_{i=1}^{f(m)} y_i(f)^*y_i(f)\| < M + \lambda/4 \tag{e 5.235}$$

for all $f \in \mathcal{F}$. This implies that

$$|\tau \circ \Lambda_1(f) - \tau \circ \Lambda_2(f)| < M + \lambda$$

for all $f \in \mathcal{F}$ and for all $\tau \in T(B)$.

□

Theorem 5.3. *Let X be a compact metric space and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing function with $\lim_{t\to 0} \Delta(t) = 0$. Let $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. Then there exists $\eta > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{H} \subset C(X)_{s.a.}$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{U} \subset U_c(K_1(C(X)))$, $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfying the following: Suppose that*

$L_1, L_2 : C(X) \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps for some unital simple C^*-algebra A of tracial rank at most one such that

$$\begin{aligned} [L_1]|_{\mathcal{P}} &= [L_2]|_{\mathcal{P}}, && \text{(e 5.236)} \\ |\tau \circ L_1(h) - \tau \circ L_2(h)| &< \gamma_1 \text{ for all } h \in \mathcal{H}, && \text{(e 5.237)} \\ \mathrm{dist}(\langle \overline{L_1(u)} \rangle, \langle \overline{L_2(u)} \rangle) &< \gamma_2 \text{ for all } u \in \mathcal{U} && \text{(e 5.238)} \\ \mu_{\tau \circ L_i}(O_r) &> \Delta(r) && \text{(e 5.239)} \end{aligned}$$

for all $\tau \in T(A)$ and for all $r \geq \eta$. Then there exists a unitary $W \in A$ such that

$$\|\mathrm{Ad}\, W \circ L_1(f) - L_2(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad \text{(e 5.240)}$$

Proof. Fix $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\eta_1 > 0$ be as in 3.6 for $\epsilon/2$ (in place of ϵ) and $\mathcal{F}$. Let $\sigma_1 = \Delta(\eta_1/3)/3$. Let $\eta_2 > 0$ be as in 3.6 for $\epsilon/2$ (in place of ϵ), $\mathcal{F}$, η_1 and σ_1. Let $\sigma_2 = \Delta(\eta_2/3)/3$. Let $\eta_3 > 0$ be as in 3.6 for $\epsilon/2$ (in place of ϵ), $\mathcal{F}$, η_1, σ_1, η_2 and σ_2. Let $\sigma_3 = \Delta(\eta_3/3)/3$. Let $\eta_4 > 0$ be as in 3.6 for $\epsilon/2$ (in place of ϵ), $\mathcal{F}$, η_1, σ_1, η_2, σ_2, η_3 and σ_3. Let $\sigma_4 = \Delta(\eta_4/3)/3$.

Let $\gamma_1' > 0$ (in place of γ_1), $\gamma_2' > 0$ (in place of γ_2), δ_1 (in place of δ), $\mathcal{G}_1 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) be a finite subset $\mathcal{H} \subset C(X)_{s.a.}$ be a finite subset and $\mathcal{U}_1 \subset U_c(K_1(C(X)))$ (in place of $\mathcal{U}$) be a finite subset as required by 3.6 for $\epsilon/2$, $\mathcal{F}$, η_i and σ_i $(i = 1, 2, 3, 4)$. Let $N \geq 1$ be an integer such that every unitary in $\mathcal{U}_1$ is in $M_N(U(C(X)))$.

Let $\Delta_1 = \Delta/2$. Let $\delta_2 > 0$ (in place of δ) and $\mathcal{G}_2 \subset C(X)$ (in place of $\mathcal{G}$) required by 3.4 for Δ_1 (in place of Δ), $\mathcal{U}_1$ (in place of $\mathcal{U}$), $\eta_4/2$ (in place η), 15/16 (in place of λ_1) and 1/32 (in place of λ_2).

Let $\delta_3 > 0$ (in place of δ), $\mathcal{G}_3 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_2 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) be a finite subset $\{x_1, x_2, ..., x_m\} \subset X$, $\mathcal{U}_2 \subset U_c(K_1(C(X))$ (in place of $\mathcal{U}$) and $K \geq 1$ (in place of L) be an integer required by 4.4 for $\epsilon/2$ (in place of ϵ), $\mathcal{F}$ and γ_2' (in place of λ).

Let $\delta_4 > 0$ (in place of δ), $\mathcal{G}_4 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset required by Lemma 5.2 for $\gamma_2'/8$ (in place of λ), $\mathcal{U}_1 \cup \mathcal{U}_2$ (in place of $\mathcal{U}$) and $\mathcal{H}$ (in place of $\mathcal{F}$).

Let $\delta_5 = \min\{\epsilon/4, \delta_i : 1 \leq i \leq 4\}$ $\mathcal{G}_5 = \mathcal{F} \cup \cup_{i=1}^4 \mathcal{G}_i$, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, $\gamma_1 = \gamma_1'/8$ and $\gamma_2 = \gamma_2/8$. Put $\epsilon_0 = \min\{\gamma_1'/8N, \gamma_2'/8N\}$ and $\eta_0 = \min\{\eta_i/4 : 1 \leq i \leq 4\}$.

Let $\eta > 0$, $\delta_6 > 0$ (in place of δ), $\mathcal{G}_6 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset required by 5.1 for δ_5 (in place of ϵ), ϵ_0, $\{x_1, x_2, ..., x_m\}$, $\mathcal{G}_5$ (in place of $\mathcal{F}$), Δ, K and η_0.

Define $\delta = \min\{\delta_6, \delta_5\}$, $\mathcal{G} = \mathcal{G}_6 \cup \mathcal{G}_5$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

Now suppose that $L_1, L_2 : C(X) \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps, where A is a unital simple C^*-algebra of tracial rank at most one, which satisfy the assumption for the above defined Δ, η, $\mathcal{H}$, $\mathcal{U}$, γ_1 and γ_2.

Let $\mathcal{H}_1 \subset A$ (in place of $\mathcal{H}$) be a finite subset and $\sigma > 0$ for L_1, L_2, $\mathcal{U}$, $\lambda_2'/8$ (in place of Γ) and $\min\{\lambda_1'/8, \lambda_2'/8\}$ (in place of λ) (for $C = C(X)$) be required by 5.2. Let $\epsilon_{00} = \sigma$.

Let $\delta_7 = \min\{\sigma/2, \delta\}$.

By applying 5.1, for $\mathcal{H}_1$ (in place of $\mathcal{H}$), there exist mutually orthogonal projections

$$P_1, P_2, P_3, p_1, p_2, ..., p_m \in A$$

with $P_2, P_3, p_1, p_2, ..., p_m \in B$, $P_1 + P_2 + P_3 + \sum_{i=1}^m p_i = 1_A$,

$$\tau(P_3) > 1 - \epsilon_0 \text{ and } K([P_1] + [P_2]) \leq [p_i], \ \ i = 1, 2, ..., m, \quad \text{(e 5.241)}$$

a unital δ_5-$\mathcal{G}_5$-multiplicative contractive completely positive linear map $\psi : C(X) \to P_2AP_2$ whose range is contained in a finite dimensional C^*-subalgebra, unital δ_5-$\mathcal{G}_5$-multiplicative contractive completely positive linear maps $H_1, H_2 : C(X) \to P_3BP_3 \subset P_3AP_3$, where $1_B = P_3$, $B = \oplus_{j=1}^N B_j$, $B_j = C(X_j, M_{r(j)})$ ($X_j = [0,1]$, or X_j is a point) with

$$[H_1]|_{\mathcal{P}} = [H_2]|_{\mathcal{P}} = [h_0]|_{\mathcal{P}} \tag{e 5.242}$$

for some unital homomorphism $h_0 : C(X) \to B$ and a unitary $W_0 \in A$ such that

$$\|[L_1(g) - [P_1L_1(g)P_1 \oplus \psi(g) \oplus \sum_{i=1}^m g(x_i)p_i \oplus H_1(g)]\| < \delta_7 \text{ and} \tag{e 5.243}$$

$$\|[\mathrm{Ad}\, W_0 \circ L_2(g) - [P_1(\mathrm{Ad}\, W_0 \circ L_2(g))P_1 \oplus \psi(g) \oplus \sum_{i=1}^m g(x_i)p_i \oplus H_2(g)]\| < \delta_7 \tag{e 5.244}$$

for all $g \in \mathcal{G}_5$,

$$\mu_{t\circ H_i}(O_r) \ge \Delta(r/3)/2 \tag{e 5.245}$$

for all $r \ge \eta_0$ and for all $t \in T(C)$, where $C = P_3BP_3$,

$$T(P_2 \oplus \sum_{i=1}^m p_i) < \epsilon_0 \tag{e 5.246}$$

for all $T \in T(B)$, and,

$$\|P_1a - aP_1\| < \epsilon_{00} \text{ and} \tag{e 5.247}$$

$$(1 - P_1)a(1 - P_1) \in_{\epsilon_{00}} B \text{ for all } a \in \mathcal{H}_1 \cup L_1(\mathcal{G}_5 \cup L_2(\mathcal{G}_5). \tag{e 5.248}$$

Moreover

$$[P_1L_1P_1]|_{\mathcal{P}} = [P_1\mathrm{Ad}\, W_0 \circ L_2P_1]|_{\mathcal{P}}. \tag{e 5.249}$$

Put $\Psi_1 = P_1L_1P_1 \oplus \psi$ and $\Psi_2 = P_1\mathrm{Ad}\, W_0 \circ L_2P_1 \oplus \psi$. By the choice of $\mathcal{H}_1$,

$$\mathrm{dist}(\overline{\langle P_1L_1P_1(u)\rangle}, \overline{\langle P_1\mathrm{Ad}\, W_0 \circ L_2P_1(u)\rangle}) < \gamma_2'/4 + \gamma_2'/2 = \lambda_2$$

for all $u \in \mathcal{U}$. Let D be a finite dimensional C^*-subalgebra of P_2AP_2 such that $\psi(C(X)) \subset D$. Then $\langle \psi(u)\rangle \in CU(P_3AP_3)$ for all $u \in \mathcal{U}$. It follows that

$$\mathrm{dist}(\overline{\langle \Psi_1(u)\rangle}, \overline{\langle \Psi_2(u)\rangle}) < \lambda_2 \tag{e 5.250}$$

for all $u \in \mathcal{U}$. By the choices of K and $\{x_1, x_2, ..., x_m\}$, there exists a unitary

$$W_1 \in (P_1 + P_2 + \sum_{i=1}^m p_i)A(P_1 + P_2 + \sum_{i=1}^m p_i)$$

such that

$$\|W_1^*(\Psi_2(f) \oplus \sum_{i=1}^m f(x_i)p_i)W_1 - \Psi_1(f) \oplus \sum_{i=1}^m f(x_i)p_i\| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \tag{e 5.251}$$

Define $\Phi_1 : C(X) \to B$ by

$$\Phi_1(f) = \psi(f) \oplus \sum_{i=1}^m f(x_i)p_i \oplus H_1(f)$$

for all $f \in C(X)$ and define $\Phi_2 : C(X) \to B$ by

$$\Phi_2(f) = \psi(f) \oplus \sum_{i=1}^{m} f(x_i)p_i \oplus H_2(f)$$

for all $f \in C(X)$.

By the choice of δ_4, $\mathcal{G}_4$ and $\mathcal{H}_1$, and applying 5.2, we obtain that

$$\mathrm{dist}(\overline{\langle \Phi_1(u) \rangle}, \overline{\langle \Phi_2(u) \rangle}) < \lambda_2'/8 \tag{e 5.252}$$

for all $u \in \mathcal{U}$ and

$$|T \circ \Phi_1(g) - T \circ \Phi_2(g)| < \lambda_1'/4 \tag{e 5.253}$$

for all $g \in \mathcal{H}$ and for all $T \in T(B)$.

Combining with (e 5.246), we obtain that

$$|t \circ H_1(f) - t \circ H_2(f)| \le \lambda_1'/4 + 2\epsilon_0 < \lambda_1' \tag{e 5.254}$$

for all $f \in \mathcal{H}$ and for all $t \in T(C)$. Using the de la Harp-Skandalis determinant, combining (e 5.252) and (e 5.246), we compute that

$$\mathrm{dist}(\overline{\langle H_1(u) \rangle}, \overline{\langle H_2(u) \rangle}) < \lambda_2'/4 + 2N\epsilon_0 < \lambda_2'. \tag{e 5.255}$$

for all $u \in \mathcal{U}$. Then, by (e 5.249) and by applying 3.6, there exists a unitary $W_2 \in C$ such that

$$\|\mathrm{Ad}\, W_2 \circ H_2(f) - H_1(f)\| < \epsilon/2 \tag{e 5.256}$$

for all $f \in \mathcal{F}$. Define $W = W_0(W_1 \oplus W_2)$. Then, by (e 5.244), (e 5.251) and (e 5.256), we finally obtain that

$$\|\mathrm{Ad}\, W \circ L_2(f) - L_1(f)\| < \epsilon$$

for all $f \in \mathcal{F}$.

□

Definition 5.4. Let X be a compact metric space and $P \in M_r(C(X))$ be a projection, where $r \ge 1$ is an integer. Put $C = PM_r(C(X))P$. Suppose $\tau \in T(C)$. It is known that there exists a probability measure μ_τ on X such that

$$\tau(f) = \int_X t_x(f(x))d\mu_\tau(x) \text{ for all } f \in C$$

where t_x is the normalized trace on $P(x)M_rP(x)$ for all $x \in X$ (see 2.17 of [13]).

Suppose that Y is a finite CW complex, $r \ge 1$ is an integer and $P \in M_r(C(Y))$ is a projection. Let $X \subset Y$ be a compact subset. Let $\pi : M_r(C(Y)) \to M_r(C(X))$ be the quotient map defined by $\pi(f) = f|_X$ for all $f \in M_r(C(Y))$.

Corollary 5.5. *Suppose that Y is a finite CW complex, $r \ge 1$ is an integer and $P \in M_r(C(Y))$ is a non-zero projection. Define $C = \pi(PM_r(C(Y))P)$ as defined above. Then Theorem 5.3 holds when $C(X)$ is replaced by C and using the measure defined in 5.4.*

Proof. Clearly the corollary holds if $C = M_r(C(X))$.

To prove the general case, we may assume that Y is connected. Then there is an integer $d \ge 1$ and a projection $Q \in M_d(PM_r(C(Y))P)$ such that $QM_r(PM_r(C(Y))P)Q \cong M_k(C(Y))$ for some integer $k \ge 1$. Put $C_1 = QM_r(PM_r(C(Y))P)Q$. Then there exists a projection $Q_1 \in M_{k_1}(C_1)$ and a unitary $W \in M_{dk_1}(PM_r(C(Y))P)$ such that $W^*Q_1W = P$. Keep the notation π as in 5.4. Note that, any unital contractive completely positive linear map $L_i : C \to A$, we obtain a unital contractive completely positive linear map $L_i \otimes \mathrm{id}_{M_d} : M_r(C) \to M_r(A)$. Put $\psi_{i,1} = (L_1 \otimes \mathrm{id}_{M_r})|_{\pi(C_1)}$ and $\psi_{i,2} = (\psi_2 \otimes \mathrm{id}_{M_{k_1}})|_{\pi(Q_1)M_{k_1}(C_1)\pi(Q_1)}$. We see that the corollary follows by first considering $\psi_{i,1}$ $(i = 1, 2)$ and then $\psi_{i,2}$ $(i = 1, 2)$. □

Definition 5.6. Let A be a unital C^*-algebra and let C be another C^*-algebra. Let $L : C \to A$ be a positive linear map. Let $\Theta : C_+ \setminus \{0\} \to \mathbb{N} \times R_+$ be a map. We write $\Theta(c) = (N(\Theta(c)), R(\Theta(c)))$ for $c \in C_+ \setminus \{0\}$, where $N(\Theta(c)) \in \mathbb{N}$ and $R(\Theta(c)) \in R_+$. Suppose that $\mathcal{S} \subset C_+$ is a subset. We say the map L is $\mathcal{S}$-Θ-full, if, for each $s \in \mathcal{S}$, there are $x_1, x_2, ..., x_{N(\Theta(s))}$ such that $\|x_j\| \le R(\Theta(s))$, $j = 1, 2, ..., N(\Theta(s))$ and

$$1_A = \sum_{j=1}^{N(\Theta(s))} x_j^* L(s) x_j. \tag{e 5.257}$$

The following is known and easy to prove. Only part (1) is actually used in this paper. Both hold for more general unital simple C^*-algebras. For example, the class of unital separable simple C^*-algebras which satisfy the strict comparison property for positive elements.

Lemma 5.7. *Let X be a compact subset of a finite CW complex Y, let $P \in M_r(C(Y))$ be a projection, where $r \ge 1$ is an integer, and let $\pi : M_r(C(Y)) \to M_r(C(X))$ be defined by $\pi(f) = f|_X$, Put $C = \pi(PM_r(C(Y))P)$.*

(1) Suppose that $\Theta : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ is a map. Then there exists an non-decreasing map $\Delta : (0, 1) \to (0, 1)$ satisfying the following: For any $\eta > 0$, there exists a finite subset $\mathcal{S} \subset C_+ \setminus \{0\}$ such that, if A is a unital separable simple C^*-algebra with $TR(A) \le 1$ and if $L : C \to A$ is a unital $\mathcal{S}$-Θ-full positive linear map, then

$$\mu_{\tau \circ L}(O_r) \ge \Delta(r) \text{ for all } \tau \in T(A)$$

for all open balls O_r with radius $r \ge \eta$.

(2) Suppose that $\Delta : (0, 1) \to (0, 1)$ is a nondecreasing map. Then there exists a map $\Theta : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ satisfying the following: For any finite subset $\mathcal{S} \subset C_+ \setminus \{0\}$, there exists $\eta > 0$ such that, if A is a unital separable simple C^*-algebra with $TR(A) \le 1$ and $L : C \to A$ is a unital positive linear map for which

$$\mu_{\tau \circ L}(O_r) \ge \Delta(r) \text{ for all } \tau \in T(A)$$

for all open balls O_r with radius $r \ge \eta$, then L is $\mathcal{S}$-Θ-full.

Theorem 5.8. *Let C be a unital AH-algebra and let $\Theta : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+$ be a map. Let $\epsilon > 0$, $\mathcal{F} \subset C$ be a finite subset. There exists a finite subset $\mathcal{S} \subset A_+ \setminus \{0\}$, $\delta > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, a finite subset $\mathcal{G} \subset C$, a finite subset $\mathcal{P} \subset \underline{K}(C)$, a finite subset $\mathcal{H} \subset A_{s.a.}$ and a finite subset $\mathcal{U} \subset U_c(K_1(C))$ satisfying the following: Suppose that A is a unital separable simple C^*-algebra with $TR(A) \le 1$ and suppose that $\varphi, \psi : C \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that φ and ψ are $\mathcal{S}$-Θ-full,*

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 5.258}$$

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \sigma_1 \text{ for all } g \in \mathcal{H}, \tag{e 5.259}$$

$$\mathrm{dist}(\overline{\langle \varphi(u) \rangle}, \overline{\langle \psi(u) \rangle}) < \sigma_2 \text{ for all } u \in \mathcal{U}. \tag{e 5.260}$$

Then there exists a unitary $w \in A$ such that

$$\|\mathrm{Ad}\, w \circ \varphi(f) - \psi(f)\| < \epsilon \ \textit{for all}\ f \in \mathcal{F}. \tag{e 5.261}$$

Proof. Let $C = \lim_{n\to\infty}(C_n, \varphi_n)$, where $C_n = P_n M_{r(n)}(C(Y_n))P_n$, X_n is a finite CW complex, $r(n) \geq 1$ is an integer, $P_n \in M_{r(n)}(C(Y_n))$ is a projection and $\varphi_n : C_n \to C_{n+1}$ is a unital homomorphism. Let $\varphi_{n,\infty} : C_n \to C$ be the unital homomorphism induced by the inductive limit system. Then, for each n, $\varphi_{n,\infty}(C_n) \cong \pi_n(P_n)M_{r(n)}(C(X_n))\pi_n(P_n)$, where $X_n \subset Y_n$ is a compact subset and $\pi_n : M_{r(n)}(C(Y_n)) \to M_{r(n)}(C(X_n))$ is defined by $\pi_n(f) = f|_{X_n}$. Let $B_n = \pi_n(P_n)M_{r(n)}(C(X_n))\pi_n(P_n)$, $n = 1, 2, ...$ Note that $B_n \subset B_{n+1}$, $n = 1, 2, ...$ We may write $C = \overline{\cup_{n=1}^{\infty} B_n}$. Let $\epsilon > 0$ and $\mathcal{F} \subset C$ be a finite subset. Without loss of generality, we may assume that $\mathcal{F} \subset B_n$ for some integer $n \geq 1$. From this it is clear that we can reduce the general case to the case that $C = B_n$. Then the result follows from 5.5 and 5.7. □

Corollary 5.9. *Let C be a unital AH-algebra and let $\Theta : C_+ \setminus \{0\} : \mathbb{N} \times \mathbb{R}_+$ be a map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\sigma_1 > 0$, $\sigma_2 > 0$, a finite subset $\mathcal{S} \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(C)$, a finite subset $\mathcal{H} \subset A_{s.a.}$ and a finite subset $\mathcal{U} \subset U_c(K_1(C))$ satisfying the following:*

Suppose that A is a unital separable simple C^*-algebra with $TR(A) \leq 1$ and suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms which are $\mathcal{S}$-Θ-full such that

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \tag{e 5.262}$$

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \sigma_1 \ \textit{for all}\ g \in \mathcal{H}\ \textit{and}\ \ \textit{for all}\ \tau \in T(A), \tag{e 5.263}$$

$$\mathrm{dist}(\varphi^{\ddagger}(\bar{u}), \psi^{\ddagger}(u)) < \sigma_2 \ \textit{for all}\ u \in \mathcal{U}. \tag{e 5.264}$$

Then there exists a unitary $w \in A$ such that

$$\|\mathrm{Ad}\, w \circ \varphi(f) - \psi(f)\| < \epsilon \ \textit{for all}\ f \in \mathcal{F}. \tag{e 5.265}$$

Theorem 5.10. *Let C be a unital AH-algebra and let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if*

$$[\varphi] = [\psi] \ \text{in}\ KL(C, A) \tag{e 5.266}$$

$$\varphi_{\sharp} = \psi_{\sharp}\ \textit{and} \tag{e 5.267}$$

$$\varphi^{\dagger} = \psi^{\dagger}. \tag{e 5.268}$$

Note that $[\varphi] = [\psi]$, $\varphi^{\dagger} = \psi^{\dagger}$ and $\varphi_{\sharp} = \psi_{\sharp}$ imply that $\varphi^{\ddagger} = \psi^{\ddagger}$. Thus Theorem 5.10 follows from 5.9 immediately.

6 The range

Definition 6.1. Let X be a compact metric space and let $C = PM_n(C(X))P$, where $P \in M_n(C(X))$ is a projection and $P(x) > 0$ for all $x \in X$, and let A be a unital separable simple C^*-algebra with $T(A) \neq \emptyset$. Let $\gamma : T(A) \to T_{\mathrm{f}}(C)$ be a continuous affine map. For any $\tau \in T(A)$ and any non-empty open set $O \subset X$, define

$$\mu_{\gamma(\tau)}(O) = \sup\{\gamma(\tau)(f) : 0 \leq f < 1 \ \text{and}\ \mathrm{supp} f \subset O\}.$$

Since $\gamma(T(A))$ is compact, we conclude that

$$\inf_{\tau \in T(A)} \mu_{\gamma(\tau)}(O) > 0$$

for every non-empty open subset $O \subset X$.

Fix $a \in (0,1)$. There are finitely many points $x_1, x_2, .., x_m \in X$ such that $\cup_{n=1}^m O(x_i, a/2) \supset X$. Let O_a be an open ball of X with center at a point x and with radius a. Then $O_a \supset O(x_i, a/2)$ for some i. Define

$$\Delta_1(a) = \min_{\{1 \le i \le m\}} \inf_{\tau \in T(A)} \mu_{\gamma(\tau)}(O(x_i, a/2)) \tag{e 6.269}$$

for all $a \in (0,1)$. It follows that

$$\mu_\tau(O_a) \ge \Delta_1(a) \text{ for all } a > 0. \tag{e 6.270}$$

Note that, if X is infinite, $\lim_{a \to 0} \Delta_1(a) = 0$.

Lemma 6.2. *Let C be as in 6.1 and A be a unital separable simple C^*-algebra with $T(A) \neq$ emptyset. Suppose that $\gamma : T(A) \to T_{\rm f}(C)$ is a continuous affine map. For any $\eta > 0$, $0 < \lambda_1, \lambda_2 < 1$, there exists a finite subset $\mathcal{H} \subset C_{s.a.}$ and $\epsilon > 0$ satisfying the following: for any unital positive linear map $L : C \to A$ such that*

$$|\tau \circ L(g) - \gamma(\tau)(g)| < \epsilon \text{ for all } g \in \mathcal{H}, \tag{e 6.271}$$

then

$$\mu_{\tau \circ L}(O_r) \ge \lambda_1 \Delta_1(a/2)/2(1+\lambda_2) \text{ for all } a \ge \eta. \tag{e 6.272}$$

The proof of this is almost identical to that of 3.4. We omit it.

Lemma 6.3. *Let X be a finite CW complex and let A be an infinite dimensional unital simple C^*-algebra with $TR(A) \le 1$. Let $C = PM_r(C(X))P$ ($r \ge 1$), where $P \in M_r(C(X))$ is a projection. Suppose that $e \in A$ is a non-zero projection. Then, there exists a non-zero projection $e_0 \le e$ and a unital monomorphism $h : C \to e_0 A e_0$.*

Proof. Without loss of generality, we may assume that X is connected. There are mutually orthogonal and mutually equivalent non-zero projections $e_1, e_2, ..., e_r \le eAe$. Put $e' = \sum_{i=1}^r e_i$. It is well known that there exists a unital monomorphism $h_0 : C(X) \to e_1 A e_1$ (see 9.5 of [15]). This extends a monomorphism $h_1 : M_r(C(X)) \to e' A e' \cong M_r(e_1 A e_1)$. Let $e_0 = h_1(P)$. Define $h : C \to e_0 A e_0$ by $h = h_1|_C$. □

Definition 6.4. Let C and A be two unital C^*-algebras. Denote by $KK_e(C,A)^{++}$ the set of those elements $\kappa \in KK(C,A)$ such that

$$\kappa([1_C]) = [1_A] \text{ and } \kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}.$$

Denote by $KL_e(C,A)^{++}$ the set of those elements $\kappa \in KL(C,A)$ such that $\kappa([1_C]) = [1_A]$ and $\kappa(K_0(C) \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$.

Now suppose that $T_{\rm f}(C) \neq \emptyset$ and A is a unital simple C^*-algebra with $T(A) \neq \emptyset$. Let $\gamma : T(A) \to T_{\rm f}(C)$ be a continuous affine map. We say κ and γ are compatible, if, $\tau \circ \kappa([p]) = \gamma(\tau)([p])$ for every projection $p \in M_\infty(C)$. Let $\alpha : U(M_\infty(C))/CU(M_\infty(C)) \to U(A)/CU(A)$ be a continuous homomorphism. By (e 2.5), there is a homomorphism $\alpha_0 : {\rm Aff}(C)/\overline{\rho_C(K_0(C))} \to {\rm Aff}(A)/\overline{\rho_A(K_0(A))}$ induced by α and there is homomorphism $\alpha_1 : K_1(C) \to K_1(A)$ induced by α. We say α and κ compatible if $\kappa|_{K_1(C)} = \alpha_1$, we say κ, γ and α are compatible if κ and γ are compatible, κ and α compatible and the homomorphism induced by γ is equal to α_0.

Lemma 6.5. *Let X be a finite CW complex, let $n \ge 1$ be an integer, let $C = PM_n(C(X))P$, where $P \in M_n(C(X))$ is a projection, and let A be a unital infinite dimensional separable simple C^*-algebra of tracial rank at most one. Suppose that $\kappa \in KK_e(C, A)^{++}$ and $\gamma : T(A) \to T_{\rm f}(C)$ is a continuous affine map which are compatible. Let $\sigma > 0$ and $\mathcal{H} \subset C_{s.a.}$ be a finite subset. Then there is a unital homomorphism $h : C \to A$ such that*

$$[h] = \kappa \text{ and} \tag{e 6.273}$$

$$|\tau \circ h(c) - \gamma(\tau)(c)| < \sigma \tag{e 6.274}$$

for all $c \in \mathcal{H}$ and all $\tau \in T(A)$.

Proof. To simplify the proof, without loss of generality, we may assume that X is connected. There is a unital separable amenable simple C^*-algebra B with $TR(B) = 0$ which satisfies the UCT such that

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Let $[\imath] \in KK_e(B, A)^{++}$ be an invertible element which gives the above identity. Therefore there is $\kappa_0 \in KK_e(C, B)^{++}$ such that

$$\kappa = \kappa_0 \times [\imath].$$

Without loss of generality, we may assume that $\mathcal{H}$ is in the unit ball of C.

Let $p \in B$ with $\tau(p) < \sigma/8$ for all $\tau \in T(B)$. It follows from 6.2 of [20] that there is a nonzero projection $p_0 \le p$, a finite dimensional C^*-subalgebra $B_0 \subset (1-p_0)B(1-p_0)$ with $1_{B_0} = 1 - p_0$, a unital homomorphism $h_1 : C \to p_0Bp_0$ and a unital homomorphism $h_2 : C \to B_0$ such that

$$[h_1 + h_2] = \kappa_0.$$

Put $D = (1 - p_0)A(1 - p_0)$. Then D is a unital simple C^*-algebra with $TR(D) \le 1$. For each $t \in T(D)$, there is a unique $\bar{t} \in T(A)$ such that $t(d) = \frac{\bar{t}(d)}{\bar{t}(1-p_0)}$ for all $d \in D$. Define $\gamma_1 : T(D) \to T_{\rm f}(C)$ by

$$\gamma_1(t) = \gamma(\bar{t}).$$

for all $t \in T(D)$. It follows from Lemma 9.5 of [15] that there exists a unital homomorphism $h_3 : C \to D$ such that

$$[h_3] = [h_2] \text{ in } KK(C, A). \tag{e 6.275}$$

$$|t(h_3(c)) - \gamma_1(t)(c)| < \sigma/8 \tag{e 6.276}$$

for all $c \in \mathcal{H}$ and for $t \in T(D)$. It follows from Theorem 5.4 of [20] that there is a unital monomorphism $j : (1 - p_0)B(1 - p_0) \to (1 - e_0)A(1 - e_0)$, where $[e_0] = [\imath]([p_0])$ such that $[j] = [\imath]$.

Now define $h : C \to A$ by $h(c) = j \circ h_1(c) \oplus h_3(c)$ for all $c \in C$. One computes that

$$[h] = [\kappa] \text{ and } |\tau(h(c)) - \gamma(\tau)(c)| < \sigma$$

for all $c \in \mathcal{H}$ and for all $\tau \in T(A)$.

□

Lemma 6.6. *Let C be as in 6.5 and let A be a unital infinite dimensional separable simple C^*-algebra with $TR(A) \le 1$. Suppose that $\kappa \in KK_e(C,A)^{++}$, $\gamma : T(A) \to T_{\rm f}(C)$ is a continuous affine map and $\alpha : U(M_\infty(C))/CU(M_\infty(C)) \to U(A)/CU(A)$ such that κ, γ and α are compatible. Then, for any $\sigma_1 > 0$, $1 > \sigma_2 > 0$, any finite subset $\mathcal{H} \subset C_{s.a.}$ and any finite subset $\mathcal{U} \subset U(M_N(C))$ (for some integer $N \ge 1$), there exists a unital homomorphism $h : C \to A$ such that*

$$[h] = \kappa, \ \ |\tau \circ h(c) - \gamma(\tau)(c)| < \sigma_1 \ \text{ for all } \ \tau \in T(A) \tag{e 6.277}$$

$$\text{and } \ \mathrm{dist}(h^{\ddagger}(\bar{u}), \alpha(\bar{u})) < \sigma_2 \ \text{ for all } \ u \in \mathcal{U}. \tag{e 6.278}$$

Proof. To simplify the notation, without loss of generality, we may assume that X is connected. Furthermore, a standard argument shows that, we can further reduce the general case to the case that $C = C(X)$.

We write $K_1(C) = G_1 \oplus Tor(K_1(C))$, where $Tor(K_1(C))$ is the torsion subgroup of $K_1(C)$ and G_1 is the free part. Fix a point $\xi_0 \in X$, define

$$C_0 = \{f \in C : f(\xi_0) = 0\}.$$

Then $C_0 \subset C$ is an ideal of C and $C/C_0 = M_r$ for some integer $r \ge 1$. We write

$$K_0(C) = \mathbb{Z}[1_C] \oplus K_0(C_0).$$

Let A_1 be a unital separable amenable simple C^*-algebra with UCT and with $TR(A_1) = TR(A) \le 1$ such that

$$(K_0(A_1), K_0(A_1)_+, [1_{A_1}], T(A_1), \rho_{A_1}) = (K_0(A), K_0(A)_+, [1_A], T(A), \rho_A) \tag{e 6.279}$$

$$\text{and } \ K_1(A_1) = G_1 \oplus Tor(K_1(A)). \tag{e 6.280}$$

To simplify notation, we may assume that $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$, where $\mathcal{U}_0 \subset U_0(M_N(C))$ and $\mathcal{U}_1 \subset U_c(M_N(C))$ are finite subsets. For each $u \in \mathcal{U}_0$, write $u = \prod_{i=1}^{n(u)} \exp(\sqrt{-1}a_i(u))$, where $a_i(u) \in M_N(C)$ is a selfadjoint elements. Write

$$a_i(u) = (a_i^{(k,j)}(u))_{N\times N}, \ \ i = 1, 2, ..., n(u).$$

Write

$$c_{i,k,j}(u) = \frac{a_i^{(k,j)} + (a_i^{(k,j)})^*}{2} \ \text{ and } \ d_{i,k,j}(u) = \frac{a_i^{(k,j)} - (a_i^{(k,j)})^*}{2i}.$$

Put

$$M = \max\{\|c\|, \|c_{i,k,j}(u)\|, \|d_{i,k,j}(u)\| : c \in \mathcal{H}, u \in \mathcal{U}_0\}.$$

Choose a non-zero projection $e \in A$ such that

$$\tau(e) < \frac{\sigma_1}{8N^2(M+1)\max\{n(u) : u \in \mathcal{U}_0\}} \ \text{ for all } \ \tau \in T(A).$$

Let $e_0 \in A_1$ be a projection such that $[e_0] = [e]$ using (e 6.279) and let $A_2 = (1 - e_0)A_1(1 - e_0)$. In what follows, we use the identification (e 6.279) Define $\theta_1 \in Hom(K_i(C), K_i(A_2))$ as follows: On $K_0(C)$, define $\theta_1(m[1_C]) = m[1 - e_0]$ for all $m \in \mathbb{Z}$, $\theta_1|_{K_0(C_0)} = \kappa|_{K_0(C_0)}$, on $K_1(C)$, define

$$\theta_1|_{Tor(K_1(C))} = \kappa|_{Tor(K_1(C))}, \ \ \theta_1|_{G_1} = \mathrm{id}|_{G_1}. \tag{e 6.281}$$

By the Universal Coefficient Theorem, there exists an element $\theta_1 \in KL(C, A_2)$ which gives the above homomorphisms. Let $\theta_2 \in KL(A_1, A)$ which gives the identification (e 6.279) and $\theta_2|_{Tor(K_1(A_1))} = \mathrm{id}|_{Tor(K_1(A_1))}$ and $\theta_2|_{G_1} = \kappa|_{G_1}$. Let $\beta = \kappa - \theta_2 \circ \theta_1$. We compute that

$$\beta([1_C]) = [e], \ \ \beta|_{K_0(C_0)} = 0, \tag{e 6.282}$$

$$\beta|_{K_1(C)} = 0. \tag{e 6.283}$$

Thus $\beta \in KK_e(C, eAe)^{++}$. It follows from 6.5 that there is a unital monomorphism $\varphi_0 : C \to eAe$ such that $[\varphi_0] = \beta$.

Choose

$$\mathcal{H}_1 = \mathcal{H} \cup \{c_{i,k,j}(u), d_{i,k,j}(u) : 1 \le k, j \le N, 1 \le i \le n(u), u \in \mathcal{U}_0\}.$$

It follows from 6.5 that there exists a unital monomorphism $\varphi_1 : C \to A_2$ such that

$$[\varphi_1] = \theta_1 \ \text{ and} \tag{e 6.284}$$

$$|\tau \circ \varphi_1(f) - \gamma(\tau)(f)| < \frac{\sigma_1}{8N^2} \tag{e 6.285}$$

for all $f \in \mathcal{H}_1$ for all $\tau \in T(A)$. Note that, for $u \in \mathcal{U}_0$,

$$\Delta(u) = \overline{\sum_{i=1}^{n(u)} \widehat{a_j(u)}}, \tag{e 6.286}$$

where $\hat{a}(\tau) = \tau(a)$ for all $a \in A_{s.a.}$. Since α and γ are compatible, we then compute that

$$\mathrm{dist}(\varphi_1^{\ddagger}(\overline{u}), \alpha(\overline{u})) < \sigma_2/8 \tag{e 6.287}$$

for all $u \in \mathcal{U}_0$. Denote by $U_c(G_1)$ the image of G_1. Define $\chi : U_c(G_1) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ by

$$\chi = \alpha|_{U_c(G_1))} - \varphi_0^{\ddagger}|_{U_c(G_1)} - \varphi_1^{\ddagger}|_{U_c(G_1)}. \tag{e 6.288}$$

Note that $U_c(G_1) \cong G_1$. We identify $U_c(G_1)$ with the corresponding part in $U_c(K_1(A_2))$. By defining χ on $Tor(K_1(A_2))$ to be zero. We obtain a homomorphism $\chi : U_c(K_1(A_2)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. It follows from Theorem 8.6 of [20] that there exists a unital homomorphism $h_1 : A_2 \to (1-e)A(1-e)$ such that

$$[h_1] = \theta_2, \ \ (h_1)_\sharp = \mathrm{id}_{T(A)} \tag{e 6.289}$$

$$\text{and } \ h_1^{\ddagger}|_{U_c(A_2)} = \chi + \theta_2|_{K_1(A_2)}, \tag{e 6.290}$$

where we identify $K_1(A_2)$ with $U_c(A_2) = U_c(G_1) \oplus Tor(K_1(A))$ and $U_c(K_1(A))$ with $K_1(A)$. We also identify $\mathrm{Aff}(T(A_2))/\overline{\rho_A(K_0(A_2))}$ with $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. Note that (by (e 6.289),

$$h_1^{\ddagger}|_{\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}} = \mathrm{id}_{\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}}.$$

Now define

$$h(f) = \varphi_0(f) \oplus h_1 \circ \varphi_1(f) \ \text{ for all } \ f \in C. \tag{e 6.291}$$

It follows that

$$[h] = \kappa, \tag{e 6.292}$$

$$|\tau \circ h(f) - \gamma(\tau)(f)| < \sigma_1 \ \text{ for all } \ f \in \mathcal{H} \ \text{ and} \tag{e 6.293}$$

$$\mathrm{dist}(h^{\ddagger}(\bar{u}), \alpha(\bar{u})) < \sigma_2 \ \text{ for all } \ u \in \mathcal{U}. \tag{e 6.294}$$

□

Lemma 6.7. *Let X be a compact subset of a finite CW complex Y. Then there exists a sequence of finite CW complex $Y_n \supset Y_{n+1}$ each of which is a compact subset of Y and there exists a contractive completely positive linear map $\varphi_n : C(X) \to C(Y_n)$ such that*

$$\pi_n \circ \varphi_n = \mathrm{id}_{C(X)}, \ \ n = 1, 2, ... \text{ and} \tag{e 6.295}$$

$$\lim_{n\to\infty} \|\varphi_n(f)\varphi_n(g) - \varphi_n(fg)\| = 0 \tag{e 6.296}$$

for all $f, g \in C(X)$, where $\pi_n : C(Y_n) \to C(X)$ is the quotient map.

Proof. Let $d_n \searrow 0$ be a decreasing sequence of positive numbers. There are finitely many open balls of Y with center in X and radius d_n covers X. Let Z_n be the union of closure of these balls. Then Z_n is a compact subset of Y which is homeomorphic to a finite CW complex. We may assume that $Z_n \supset Z_{n+1}$. Then (by, for example, The Effros-Choi Theorem), there exists, for each n, a contractive completely positive linear map $\psi_n : C(X) \to C(Z_n)$ such that

$$\pi_n \circ \psi_n = \mathrm{id}_{C(X)}, \tag{e 6.297}$$

where $\pi_n(f) = f|_X$ for $f \in C(Z_n)$, $n = 1, 2, ...$

Let $\{\mathcal{F}_m\} \subset C(X)$ be a sequence of increasing finite subsets of the unit ball of $C(X)$ so that its union is dense in the unit ball of $C(X)$. Choose $Y_1 = Z_1$ and $\varphi_1 = \psi_1$. Let $\mathcal{G}_1 = \mathcal{F}_1 \cup \{fg : f, g \in \mathcal{F}_1\}$. Choose d_{n_2} such that

$$|\psi_1(f)(x) - \psi_1(f)(x')| < 1/4 \ \text{ for all } \ f \in \mathcal{G}_1, \tag{e 6.298}$$

provided that $\mathrm{dist}(x, x') < d_{n_2}$ for all $x, x' \in Z_1$. By (e 6.297),

$$\psi_1(fg)(x) - \psi_1(f)(x)\psi_1(g)(x) = 0 \tag{e 6.299}$$

for all $x \in X$. Now for any $z \in X_{n_2}$, there exists $x \in X$ such that $\mathrm{dist}(x, z) < d_{n_1}$. Therefore, by (e 6.299) and (e 6.298),

$$\|\psi_1(fg)(z) - \psi_1(f)(z)\psi_1(g)(z)| \le \|\psi_1(fg)(z) - \psi_1(fg)(x)\| \tag{e 6.300}$$

$$+\|\psi_1(fg)(x) - \psi_1(f)(x)\psi_1(g)(x)\| \tag{e 6.301}$$

$$+\|\psi_1(f)(x)\psi_1(g)(x) - \psi_1(f)(z)\psi_1(g)(z)\| < 3/4 \tag{e 6.302}$$

for all $f, g \in \mathcal{F}_1$. Choose $Y_2 = Z_{n_2}$. Define $h_1 : C(Z_1) \to C(Z_{n_2})$ defined by $h_1(f) = f|_{Z_{n_2}}$ for all $f \in C(Z_1)$. Define $\varphi_2 : C(X) \to C(Y_2)$ by defining

$$\varphi_2(f) = h_1 \circ \psi_1.$$

Thus, by (e 6.300),

$$\|\varphi_2(fg) - \varphi_2(f)\varphi_2(g)\| < 3/4 \tag{e 6.303}$$

for all $f, g \in \mathcal{F}_1$. Note that

$$\pi_{n_2} \circ \varphi_2 = \mathrm{id}_{C(X)}. \tag{e 6.304}$$

Let $\mathcal{G}_2 = \mathcal{G}_1 \cup \mathcal{F}_2 \cup \{fg : f, g \in \mathcal{F}_2\}$. Choose d_{n_3} such that

$$|\varphi_2(f)(x) - \varphi_2(f)(x')| < 1/4^2 \ \text{ for all } \ f \in \mathcal{G}_2, \tag{e 6.305}$$

provided that $\mathrm{dist}(x, x') < d_{n_3}$ for all $x, x' \in Y_2$. By (e 6.297), for any $x \in X$,

$$\varphi_2(fg)(x) = \varphi_2(f)(x)\varphi_2(g)(x) \tag{e 6.306}$$

Then, for any $z \in Z_{n_3}$, there exists $x \in X$ such that $\mathrm{dist}(x,z) < d_{n_3}$. Thus, by (e 6.306) and (e 6.305),

$$|\varphi_2(fg)(z) - \varphi_2(f)(z)\varphi_(g)(z)| < 3/4^2 \text{ for all } z \in Z_{n_3}. \tag{e 6.307}$$

Let $h_2 : C(Y_2) \to C(Z_{n_3})$ be defined by $h_2(f) = f|_{Z_{n_3}}$ for all $f \in C(Y_2)$. Put $Y_3 = Z_{n_3}$. Define $\varphi_3 : C(X) \to C(Y_{n_3})$ by $\varphi_3(f) = h_2 \circ \varphi_2$. Then

$$\pi_{n_3} \circ \varphi_3 = \mathrm{id}_{C(X)}. \tag{e 6.308}$$

By (e 6.307), we have that

$$\|\varphi_3(fg) - \varphi_3(f)\varphi_3(g)\| < 3/4^2 \tag{e 6.309}$$

for all $f, g \in \mathcal{F}_2$. In this fashion, we obtain a sequence of contractive completely positive linear map $\varphi_k : C(X) \to C(Y_k)$, where $Y_k = Z_{n_k}$, such that

$$\pi_{n_k} \circ \varphi_k = \mathrm{id}_{C(X)} \text{ and} \tag{e 6.310}$$

$$\|\varphi_k(fg) - \varphi_k(f)\varphi_k(g)\| < 3/4^{k-1}, \tag{e 6.311}$$

for all $f, g \in \mathcal{F}_k$, $k = 1, 2, ...$ It follows that, for any $f, g \in C(X)$,

$$\lim_{k\to\infty} \|\varphi_k(fg) - \varphi_k(f)\varphi_k(g)\| = 0. \tag{e 6.312}$$

□

We have the following corollary.

Corollary 6.8. *Let Y be a finite CW complex and $P \in M_r(C(Y))$ be a non-zero projection for some integer $r \geq 1$. Let X be a compact metric space of Y and let $C = \pi(PM_r(C(Y))P)$, where $\pi : M_r(C(Y)) \to M_r(C(X))$ be the quotient map defined by $\pi(f) = f|_X$. Then there exists a sequence of finite CW complex $Y \supset Y_n \supset Y_{n+1}$ each of which is a compact subset of Y and there exists a contractive completely positive linear map $\varphi_n : C \to P_n(C(Y_n))P_n$ such that*

$$\pi_n \circ \varphi_n = \mathrm{id}_C, \ \ n = 1, 2, ... \text{ and} \tag{e 6.313}$$

$$\lim_{n\to\infty} \|\varphi_n(f)\varphi_n(g) - \varphi_n(fg)\| = 0 \tag{e 6.314}$$

for all $f, g \in C(X)$, where $P_n = P|_{Y_n}$ and $\pi_n : C(Y_n) \to C(X)$ is the quotient map defined by $\pi_n(f) = f|_X$ for all $f \in C(Y_n)$.

Lemma 6.9. *Let Y be a finite CW complex and $P \in M_r(C(Y))$ be a non-zero projection for some integer $r \geq 1$. Let X be a compact metric space of Y and let $C = \pi(PM_r(C(Y))P)$, where $\pi : M_r(C(Y)) \to M_r(C(X))$ is the quotient map defined by $\pi(f) = f|_X$. Suppose that A is a unital infinite dimensional separable simple C^*-algebra with $TR(A) \leq 1$. For any $\kappa \in KL_e(C, A)^{++}$, any affine continuous map $\gamma : T(A) \to T_{\mathrm{f}}(C)$ and any continuous homomorphism $\alpha : U(M_\infty(C))/CU(M_\infty(C)) \to U(A)/CU(A)$ such that κ, γ and α are compatible, then there is a unital homomorphism $h : C \to A$ such that*

$$[h] = \kappa, \ \ h_\sharp = \gamma \text{ and } h^\ddagger = \alpha. \tag{e 6.315}$$

Proof. Let Y_n, P_n, π_n and φ_n be as given by 6.8. Let $B_n = P_nM_r(C(Y_n))P_n$. Let $\{q_n'\}$ be a sequence of non-zero projections such that $\tau(q_n') < 1/n$ for all $\tau \in T(A)$, $n = 1, 2, ...$ Since $q_n'Aq_n'$ is a unital infinite dimensional simple C^*-algebra of tracial rank at most one, by 6.3, there exists a non-zero projection $q_n \le q_n'$ and a unital monomorphism $\varphi_{0,n} : B_n \to q_nAq_n$, $n = 1, 2, ...$

Define $\gamma_n : T(A) \to T_{\rm f}(B_n)$ by

$$\gamma_1(\tau)(b) = \tau(1 - q_n)\gamma(\tau)(\pi_n(b)) + \tau \circ \varphi_{0,n}(b) \tag{e 6.316}$$

for all $b \in B_n$ and for all $\tau \in T(A)$.

Define $s_n : B_n \to B_{n+1}$ by $s_n(f) = f|_{Y_n}$ for all $f \in B_n$, $n = 1, 2, ...$

Let $\kappa_n = \kappa \circ [\pi_n]$ be in $Hom_\Lambda(\underline{K}(B_n), \underline{K}(A))$ and let $\alpha_n : U(M_\infty(B_n))/CU(M_\infty(C)) \to U(A)/CU(A)$ defined by $\alpha_n = \alpha \circ \pi_n^\ddagger$.

Note that $K_i(B_n)$ is finitely generated ($i = 0, 1$). Let $\epsilon_n > 0$, $\mathcal{F}_n \subset B_n$ be a finite subset and $Q_n \subset \underline{K}(B_n)$ be a finite subset such that $(\epsilon_n, \mathcal{F}_n, \mathcal{P}_n)$ is a $\underline{K}$-triple and $(\epsilon_n, \mathcal{F}_n)$ is KK-pair for B_n, $n = 1, 2, ...$ Put $\mathcal{Q}_n = [\pi_n](\mathcal{P}_n)$, $n = 1, 2, ...$ We may assume that $[s_n](\mathcal{P}_n) \subset \mathcal{P}_{n+1}$ and $\cup_{n=1}^\infty \mathcal{Q}_n = \underline{K}(C)$. Let $\mathcal{G}_n \subset C$ be a finite subset and let $\delta_n > 0$ such that $(\delta_n, \mathcal{G}_n, \mathcal{Q}_n)$ is a $\underline{K}$-triple.

Choose, for each n, a finite subset $\mathcal{F}_n \subset B_n$ such that $s_n(\mathcal{F}_n) \subset \mathcal{F}_{n+1}$ and $\cup_{n=1}^\infty \pi_n(\mathcal{F}_n)$ is dense in C. Choose, for each n, a finite subset $\mathcal{H}_n \subset (B_n)_{s.a}$ such that $s_n(\mathcal{H}_n) \subset \mathcal{H}_{n+1}$ and $\cup_{n=1}^\infty \pi_n(\mathcal{H}_n)$ is dense in $C_{s.a.}$. Choose, for each n, a finite subset $\mathcal{U}_n \subset U(M_{N(n)}(B_n))$ (for some integer $N(n)$) such that $s_n(\mathcal{U}_n) \subset \mathcal{U}_{n+1}$ and $\cup_{n=1}^\infty \pi_n(\mathcal{U}_n)$ is dense in $U(M_\infty(C))$.

It follows from 6.6 there is, for each n, a unital monomorphism $h_n : B_n \to A$ such that

$$[h_n] = \kappa_n \tag{e 6.317}$$

$$|\tau \circ h_n(f) - \gamma_n(\tau)(f)| < 1/2^n \text{ for all } f \in \mathcal{H}_n \text{ and for all } \tau \in T(A) \tag{e 6.318}$$

$$\text{and } \mathrm{dist}(h_n^\ddagger(\bar{u}), \alpha_n(\bar{u})) < 1/2^n \text{ for all } u \in \mathcal{U}_n, \tag{e 6.319}$$

$n = 1, 2, ...$ Define $L_n = h_n \circ \varphi_n$. Note that

$$\pi_n \circ \varphi_n = \mathrm{id}_C \text{ and} \tag{e 6.320}$$

$$\lim_{n\to\infty} \|\varphi_n(fg) - \varphi_n(f)\varphi_n(g)\| = 0 \text{ for all } f, g \in C. \tag{e 6.321}$$

Thus, without loss of generality, we may assume that φ_n is $(\delta_{n-1}, \mathcal{G}_{n-1})$-multiplicative. It then follows that from (e 6.320) and (e 6.317) that

$$[h_n \circ \varphi_n]|_{\mathcal{Q}_n} = (\kappa_n)|_{\mathcal{Q}_n}, \ \ n = 1, 2, ... \tag{e 6.322}$$

By (e 6.322), (e 6.318), (e 6.319), combining 6.2 and applying 5.5, we obtain a subsequence $h_{n_k} \circ \varphi_{n_k} : C \to A$ and a sequence of unitaries $\{u_k\} \subset A$ such that

$$\|\mathrm{Ad}\, u_k \circ h_{n_{k+1}} \circ \varphi_{n_{k+1}}(f) - \mathrm{Ad}\, u_{k-1} \circ h_{n_k} \circ \varphi_{n_k}(f)\| < 1/2^{k+1} \tag{e 6.323}$$

for all $f \in \mathcal{F}_k$. It follows that $\{\mathrm{Ad}\, u_{k-1} \circ h_{n_k} \circ \varphi_{n_k}(f)\}$ is Cauchy for all $f \in C$. Define $h : C \to A$ by

$$h(f) = \lim_{k\to\infty} \mathrm{Ad}\, u_{k-1} \circ h_{n_k} \circ \varphi_{n_k}(f) \text{ for all } f \in C.$$

It is ready to check that h satisfy all requirements of the lemma.

□

Theorem 6.10. *Let C be a unital AH-algebra and let A be a unital infinite dimensional separable simple C^*-algebra with $TR(A) \le 1$. For any $\kappa \in KL_e(C,A)^{++}$, any affine continuous map $\gamma : T(A) \to T_{\rm f}(C)$ and any continuous homomorphism $\alpha : U(M_\infty(C))/CU(M_\infty(C)) \to U(A)/CU(A)$ such that κ, γ and α are compatible, then there is a unital homomorphism $h : C \to A$ such that*

$$[h] = \kappa, \ \ h_\sharp = \gamma \ \text{ and } \ h^\ddagger = \alpha. \tag{e 6.324}$$

Proof. Write $C = \lim_{n\to\infty}(B_n, \psi_n)$, where $B_n = P_n M_{r(n)}(C(Y_n))P_n$, Y_n is a finite CW complex, $P_n \in M_{r(n)}(C(Y_n))$ is a projection and $\psi_n : B_n \to B_{n+1}$ is a unital homomorphism. Denote by $\psi_{n,\infty} : B_n \to C$ the unital homomorphism induced by the inductive limit system. Then $\varphi_{n,\infty}(B_n) \cong Q_n M_{r(n)}(C(X_n))Q_n$, where $X_n \subset Y_n$ is a compact subset, $Q_n = \pi_n(P_n)$ and where $\pi_n : M_{r(n)}(C(Y_n)) \to M_{r(n)}(C(X_n))$ is the quotient map defined by $\pi_n(f) = f|_{X_n}$. Put $C_n = \varphi_{n,\infty}(B_n)$. We will identify C_n with $Q_n M_{r(n)}(C(X_n))Q_n$ and write $C = \overline{\cup_{n=1}^\infty C_n}$, where $C_n = Q_n M_{r(n)}(C(X_n)Q_n$,

Denote by $\imath_n : C_n \to C_{n+1}$ and $\imath_{n,\infty} : C_n \to C$ be the embedding, respectively. Let $\kappa_n = \kappa \circ [\imath_{n,\infty}]$ be in $Hom_\Lambda(\underline{K}(C_n), \underline{K}(A))$ and let $\alpha_n : U(M_\infty(C_n))/CU(M_\infty(C)) \to U(A)/CU(A)$ be defined by $\alpha_n = \alpha \circ \imath_{n,\infty}^\ddagger$. Let $(\imath_{n,\infty})_\sharp : T_{\rm f}(C) \to T_{\rm f}(C_n)$ induced by $\imath_{n,\infty}$ and define $\gamma_n : T(A) \to T_f(C_n)$ by $(\imath_{n,\infty})_\sharp \circ \gamma$. Put $\mathcal{Q}_n = [\imath_{n,\infty}](\mathcal{P}_n)$, $n = 1, 2, ...$ We may assume that $\cup_{n=1}^\infty \mathcal{Q}_n = \underline{K}(C)$. Choose, for each n, a finite subset $\mathcal{F}_n \subset C_n$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\cup_{n=1}^\infty \mathcal{F}_n$ is dense in C. It follows from 6.9 there is, for each n, a unital monomorphism $\varphi_n : C_n \to A$ such that

$$[\varphi_n] = \kappa_n, \ \ (\varphi_n)_\sharp = \gamma_n \ \text{ and } \ \varphi_n^\ddagger = \alpha_n \tag{e 6.325}$$

$n = 1, 2, ...$ By applying 5.10, for each n, there exists a unitary $u_n \in A$ (with $u_0 = 1$) such that

$$\|\mathrm{Ad}\, u_n \circ \varphi_{n+1} \circ \imath_n(f) - \mathrm{Ad}\, u_{n-1} \circ \varphi_n(f)\| < 1/2^n \ \text{ for all } \ f \in \mathcal{F}_n, \tag{e 6.326}$$

$n = 1, 2, ...$ We obtain a unital monomorphism $h : C \to A$ such that

$$h(f) = \lim_{n\to\infty} \mathrm{Ad}\, u_n \circ \varphi_{n+1} \circ \imath_n(f) \ \text{ for all } \ f \in C. \tag{e 6.327}$$

One checks that h meets all requirements of the theorem.

□

Corollary 6.11. *Let C be a unital AH-algebra and let A be a unital infinite dimensional separable simple C^*-algebra with $TR(A) \le 1$. For any $\kappa \in KL_e(C,A)^{++}$, any affine continuous map $\gamma : T(A) \to T_{\rm f}(C)$ and any continuous homomorphism $\alpha : K_1(C) \to \mathrm{Aff}(T(A))/\overline{K_0(A)}$ such that κ, γ are compatible, then there is a unital homomorphism $h : C \to A$ such that*

$$[h] = \kappa, \ \ h_\sharp = \gamma \ \text{ and } \ h^\dagger = \alpha. \tag{e 6.328}$$

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